Day 3 (LAST!) of a boring lecture series

Carl Bender

Washington University

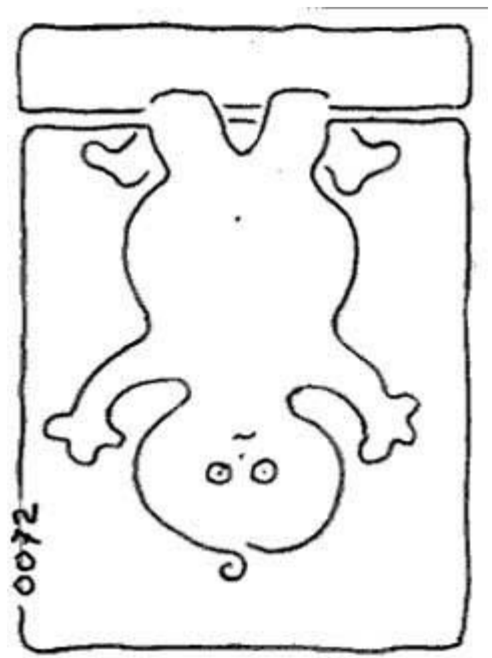
DID WE *PROVE* THAT THE EIGENVALUES ARE REAL?

NO!!

Why not??

Stability of upside-down potentials

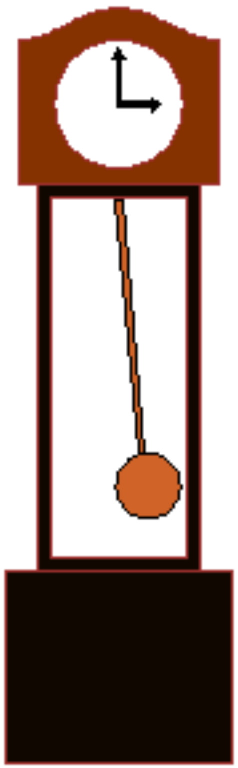
$$V(x) = -x^4$$



This upside-down potential looks *unstable* (*<u>on the real axis</u>*)

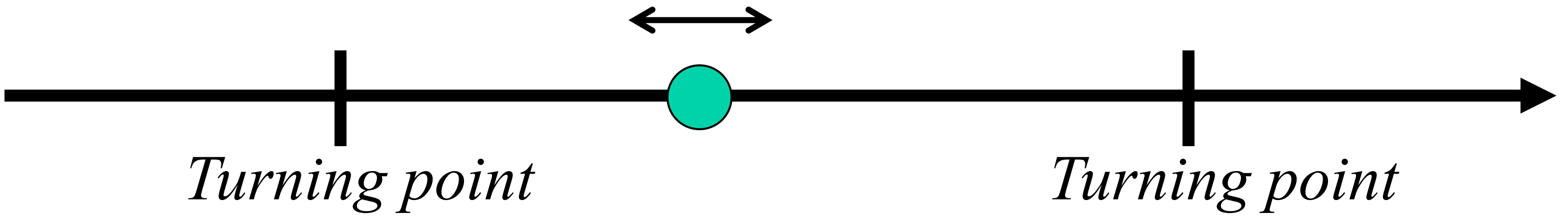
Complex variables explains why it has quantum <u>bound states</u>

To begin, we extend simple classical harmonic motion to the complex domain…



Classical harmonic oscillator

Back and forth motion on the real-x axis:



$$E = p^2 + x^2$$

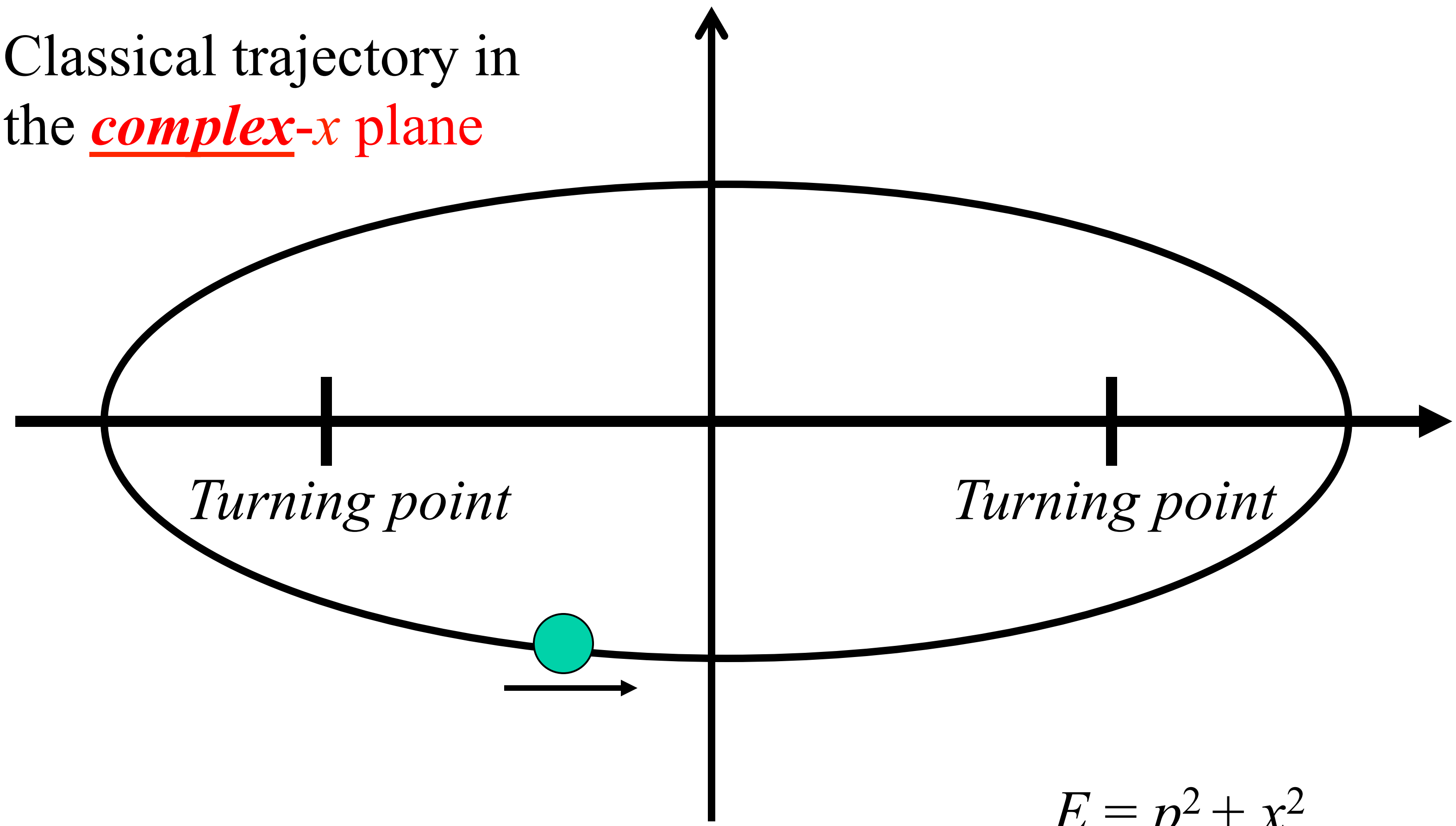
Classically allowed and ***classically forbidden*** regions…

Classically allowed and *classically forbidden* regions

Classical harmonic oscillator in the complex plane

$$H = p^2 + x^2$$ $(\varepsilon = 0)$

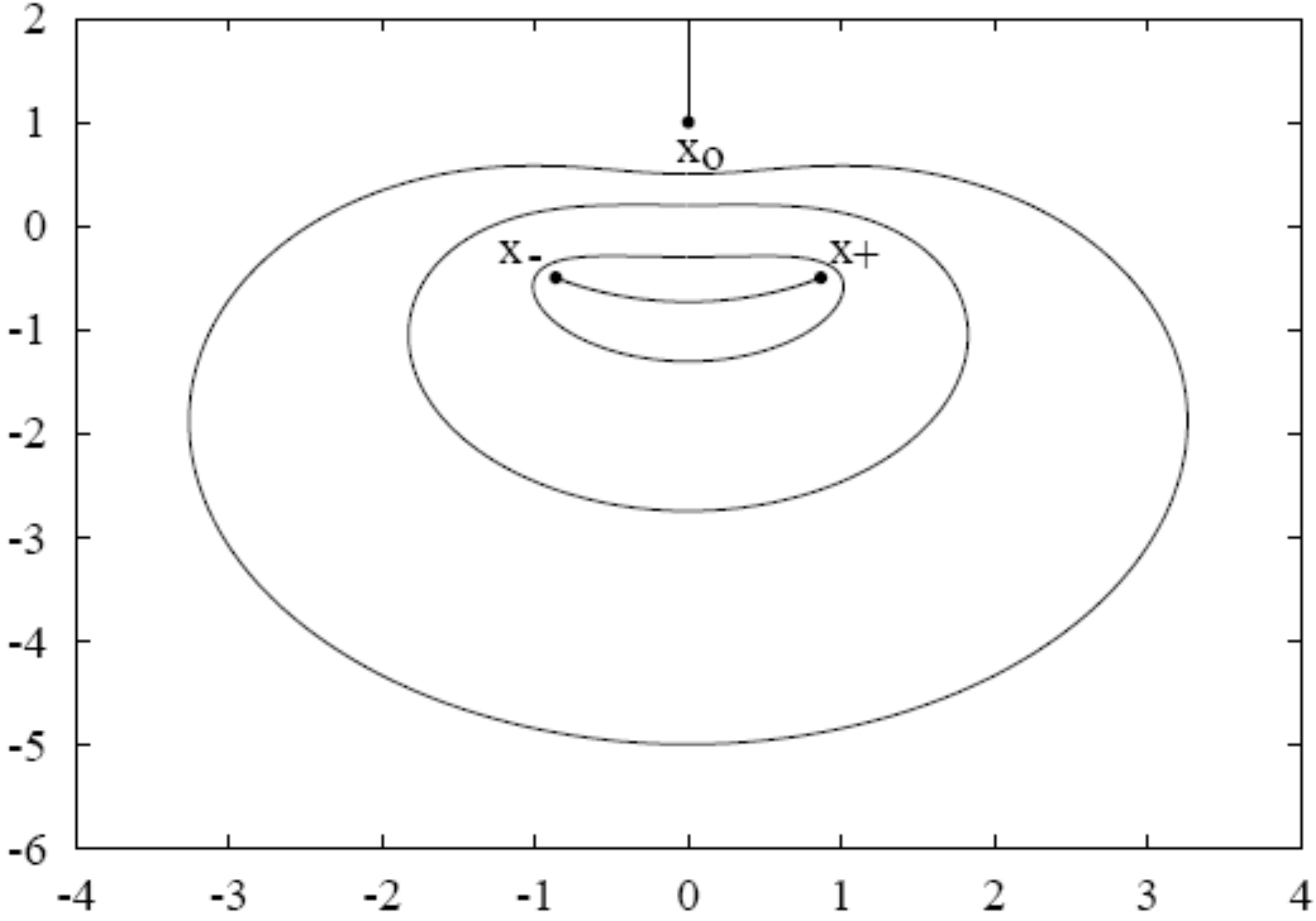
Classical trajectory in the ***complex***-x plane



$$E = p^2 + x^2$$

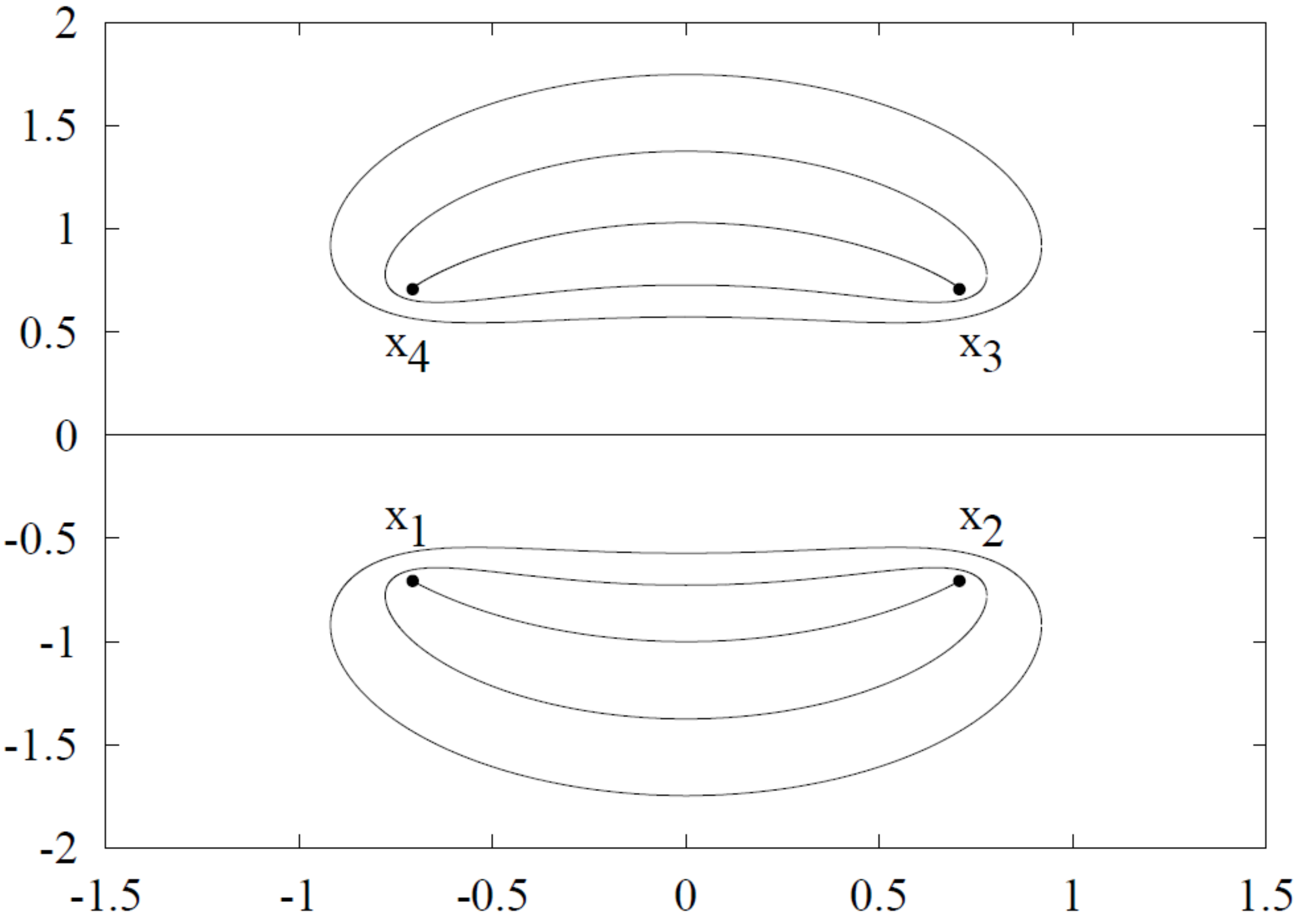
$H = p^2 + ix^3$ ($\varepsilon = 1$)

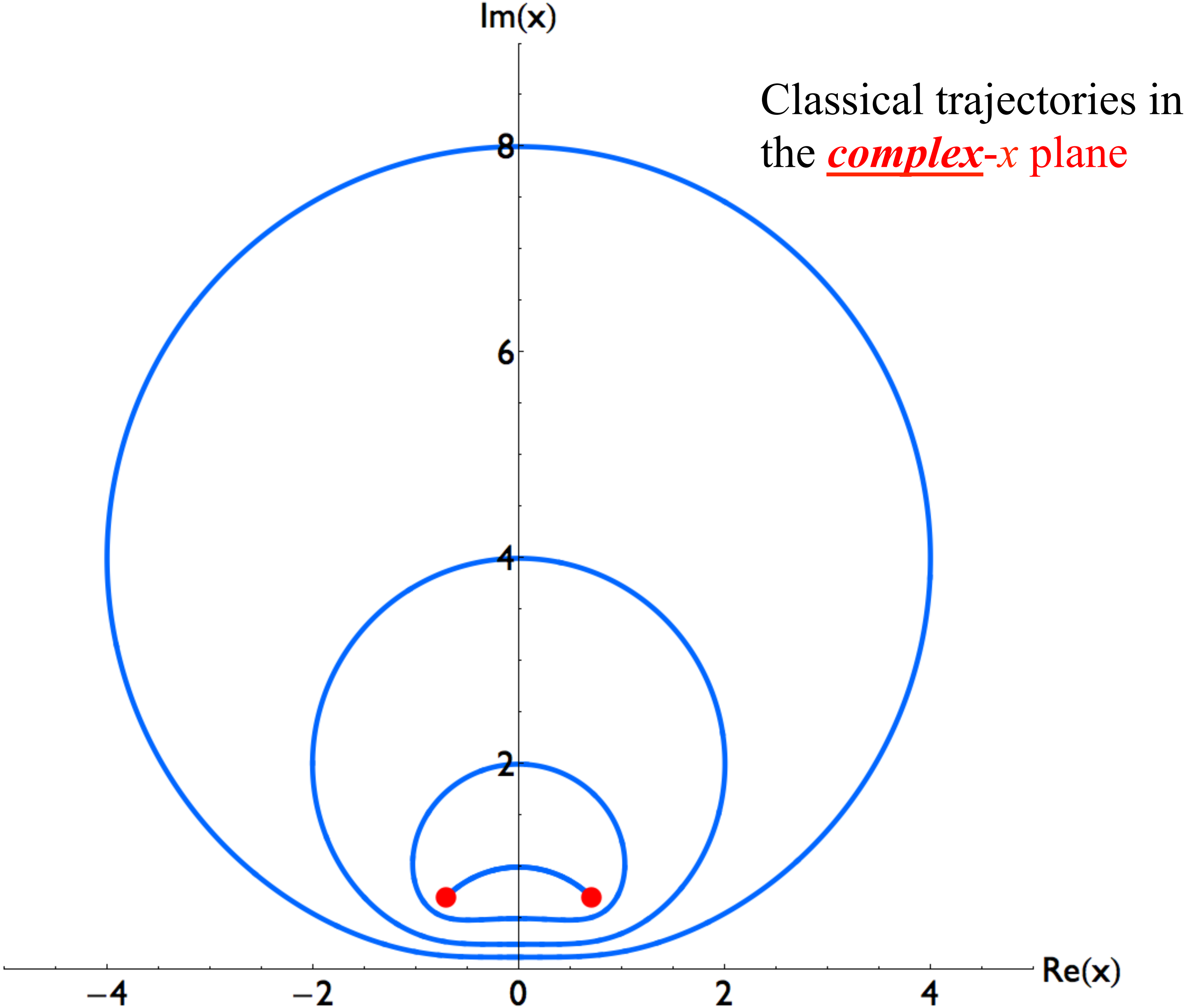
Classical trajectories in the ***complex***-*x* plane



$H = p^2 - x^4$ $(\varepsilon = 2)$

Classical trajectories in the ***complex***-*x* plane

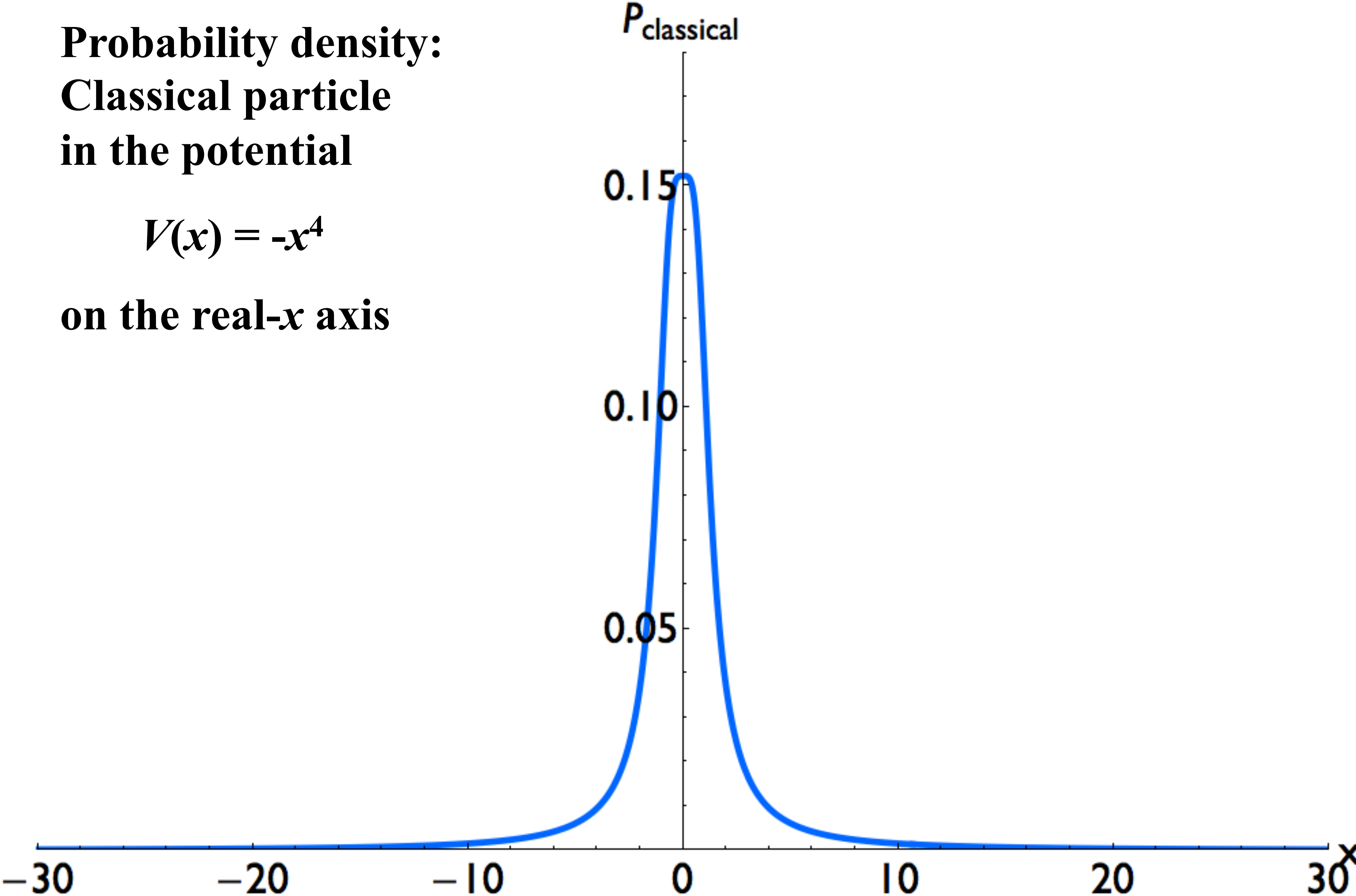


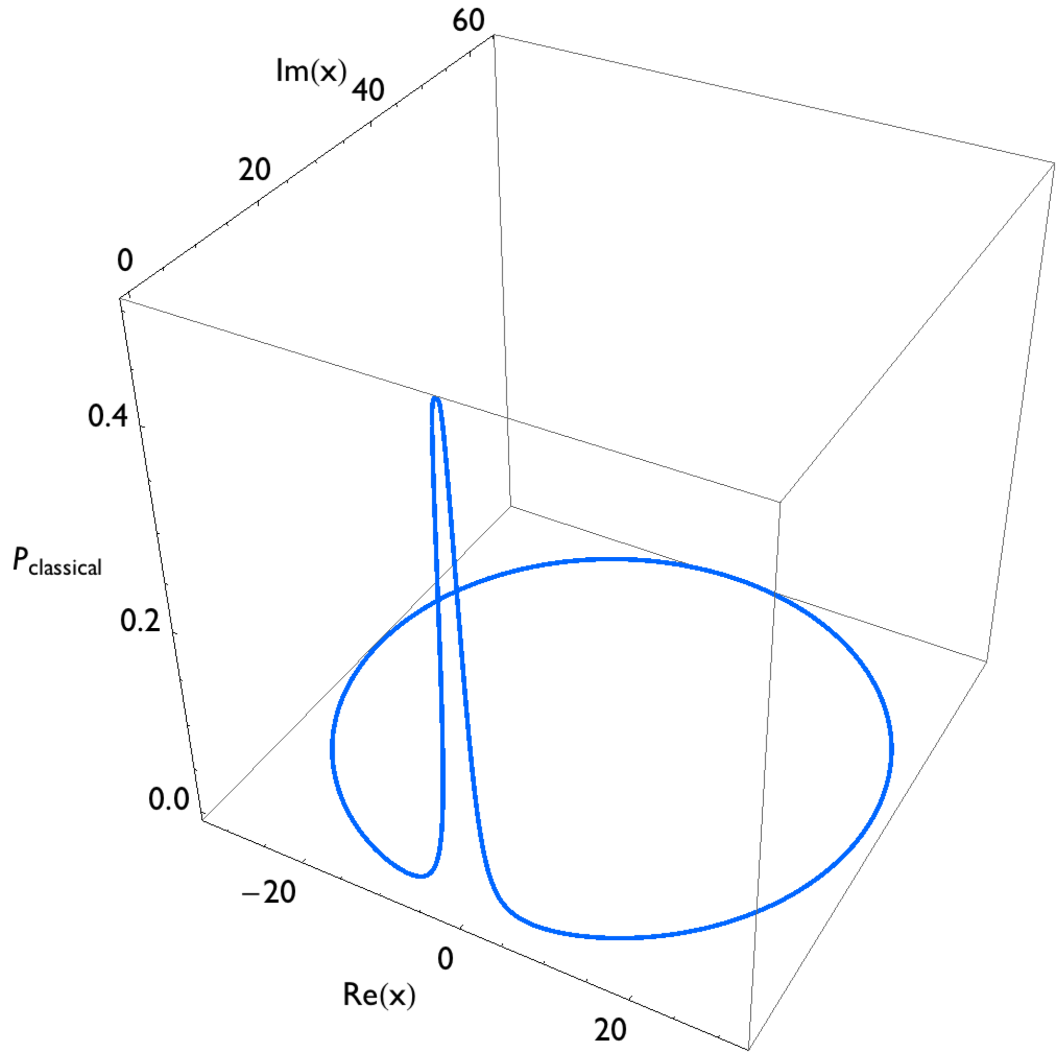

Im(x)
8
6
4
2
Re(x)
−4
−2
0
2
4
Classical trajectories in the *complex-x* plane

Probability density: Classical particle in the potential

$$V(x) = -x^4$$

on the real-*x* axis





System with a static instability becomes dynamically stable in the complex domain!

Bohr-Sommerfeld
Quantization of a complex atom

$$\oint dx\, p = \left(n + \tfrac{1}{2}\right)\pi$$

Instability at $x = 0$ is tamed!

Complex analysis enables one to *tame instabilities*!

Physical systems that seem to be unstable can become *stable* in the complex domain!

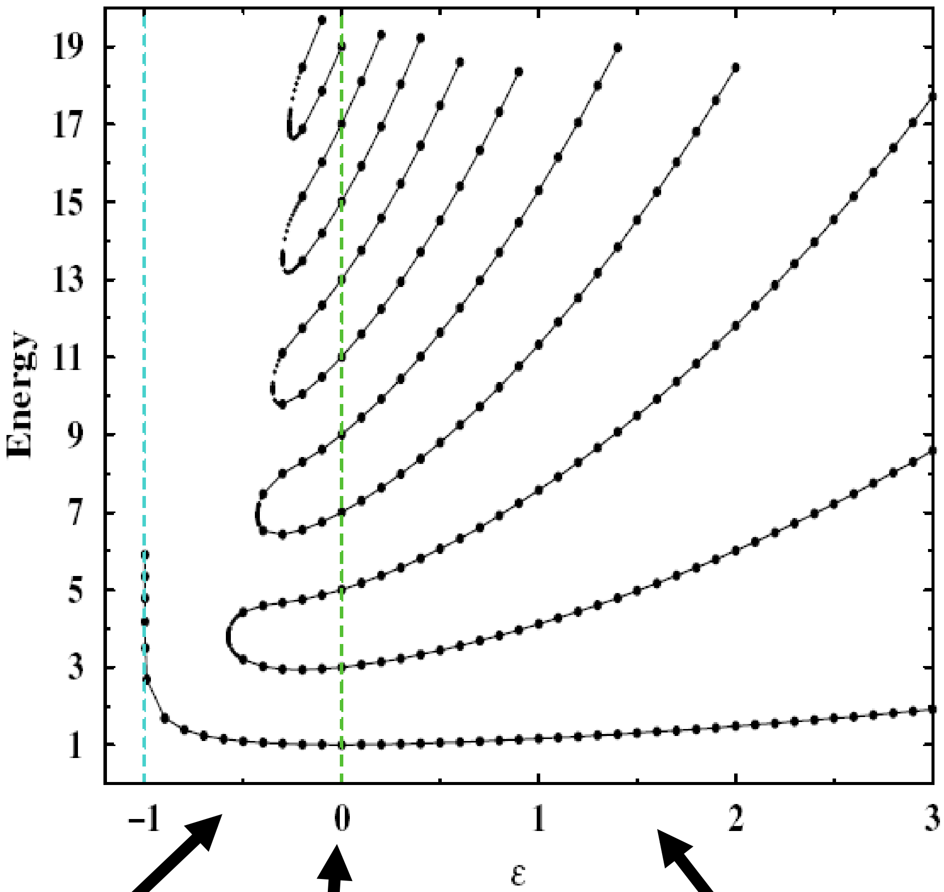
THE BASIC REASON:

If you extend real numbers to complex numbers, you lose the *ordering* property of real numbers

You lose the concept of > and <

Physical systems that *look* unstable may not be!
(examples: bicycles, tops, …)

$$H = p^2 + x^2(ix)^\varepsilon \quad (\varepsilon \text{ real})$$



Transition

at ε = 0

Region of ***broken*** ***PT*** symmetry

PT **Boundary**

Region of ***unbroken*** ***PT*** symmetry

PT **symmetry does not conflict with conventional quantum theory, but it is *weaker* than Hermiticity:**
All eigenvalues E of a Hermitian Hamiltonian are real.
But for *PT*-symmetric Hamiltonians
only the secular equation $\det(H - IE) = 0$ ***is real.***

For non-Hermitian *PT*-symmetric Hamiltonians, there are *TWO* possibilities:

***PT*-symmetric theories may have an *all* real or a *partly* real and partly complex spectrum.**

Broken ***P***arro***T*** — ***Complex spectrum***

Unbroken ***P***arro***T*** — ***Real spectrum***

Hermitian Hamiltonians: BORING!

Eigenvalues are always real – nothing interesting happens

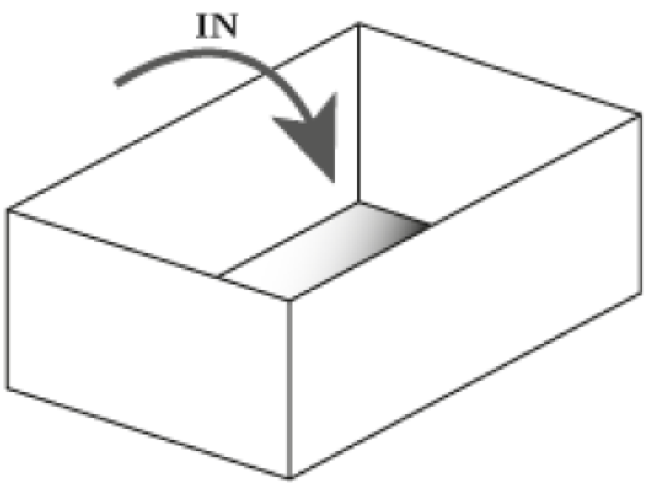


PT-symmetric Hamiltonians: ASTONISHING!

Transition between parametric regions of broken and unbroken *PT* symmetry -- can be observed experimentally!

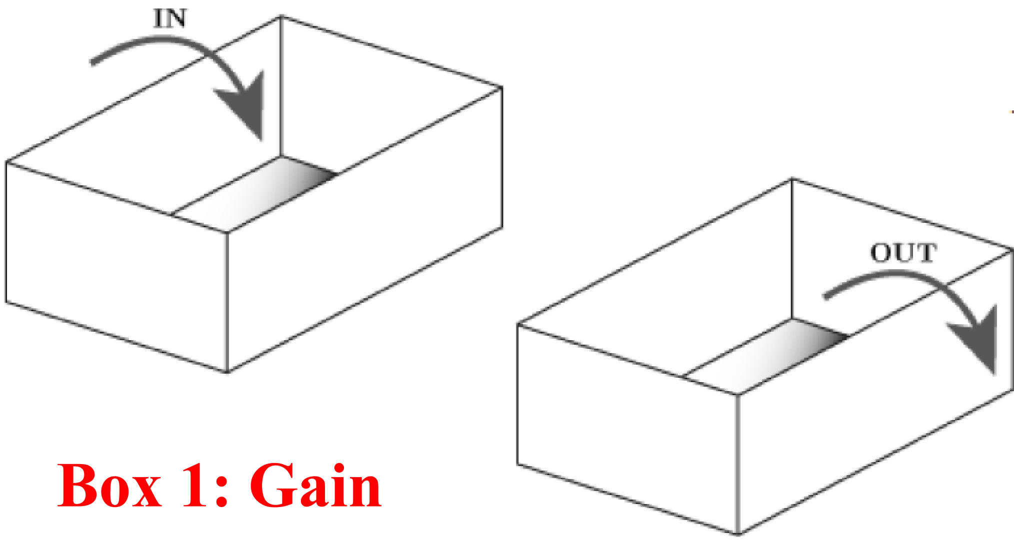
Intuitive explanation of the *PT* transition …

Imagine a closed box with gain… Hamiltonian for this system is non-Hermitian: $H = [a+ib]$



Box 1: Gain

Two noninteracting closed boxes, one with gain, the other with loss:



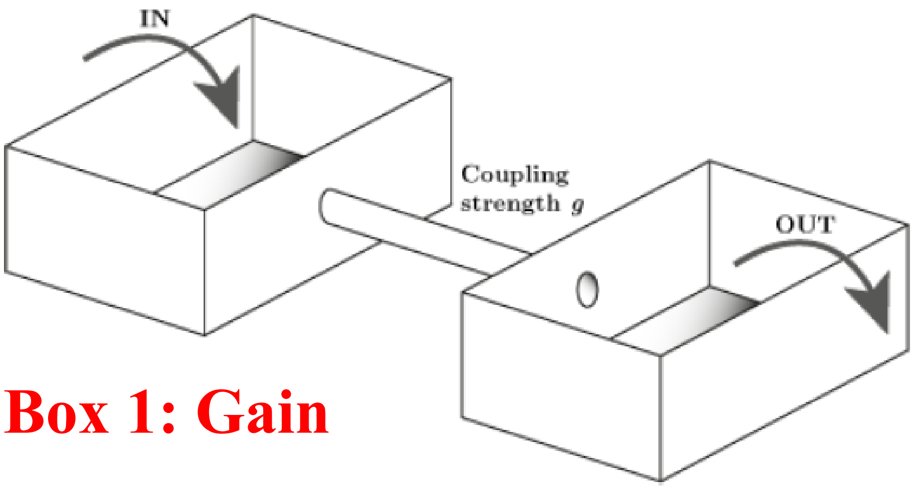
$$H_{\text{combined}} = \begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix}$$

Box 1: Gain

Box 2: Loss

This system is *not in equilibrium*

Couple the boxes:



Box 1: Gain

Box 2: Loss

$$H_{\text{coupled}} = \begin{bmatrix} a+ib & g \\ g & a-ib \end{bmatrix}$$

This Hamiltonian is non-Hermitian but *PT* symmetric:

Time reversal: $\mathcal{T}$ = complex conjugation

Parity: $\mathcal{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Real secular equation:

$$\det\left(H_{\text{coupled}} - IE\right) = E^2 - 2aE + a^2 + b^2 - g^2$$

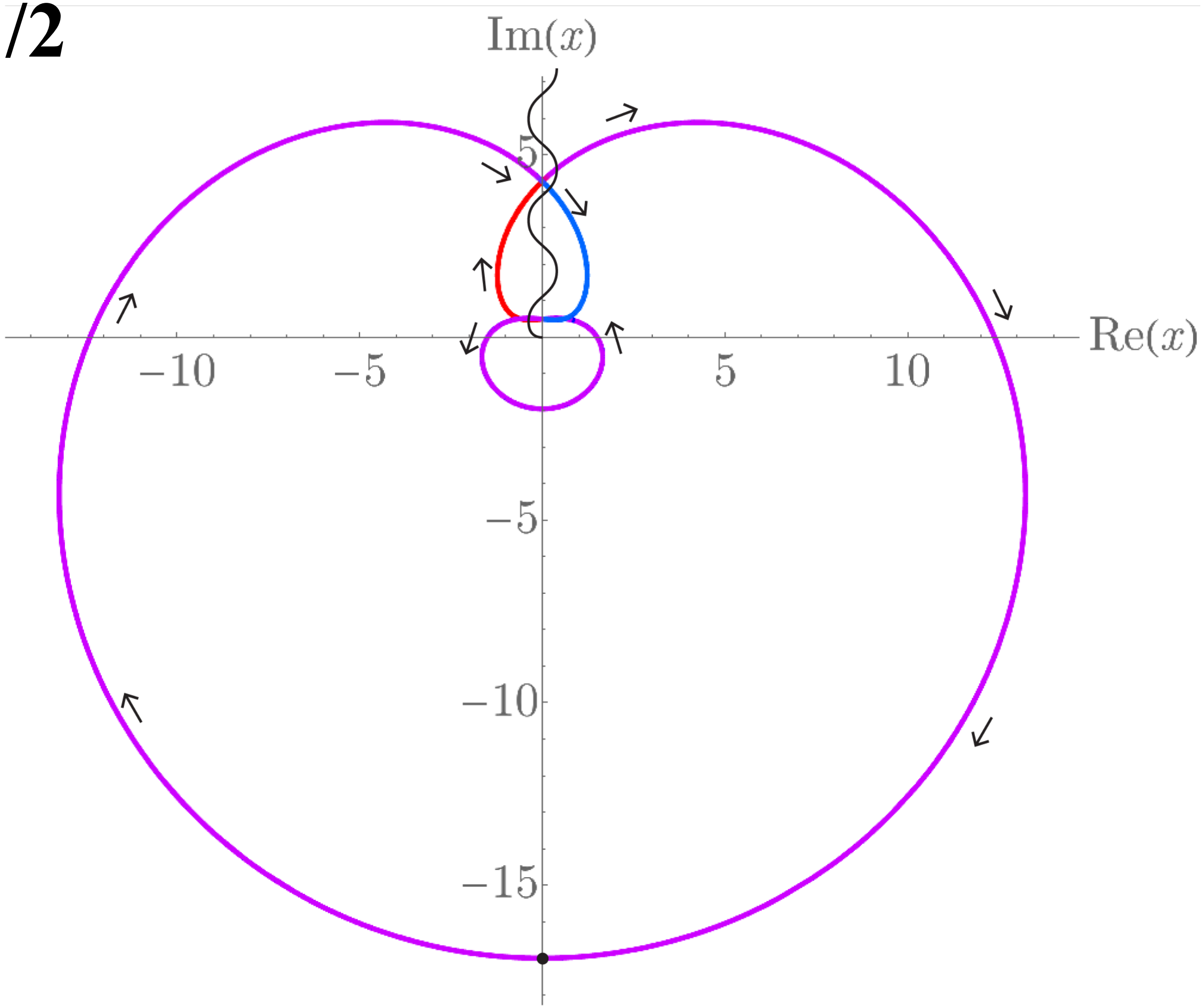
$$E_{\pm} = a \pm \sqrt{g^2 - b^2}$$

Transition at $|g| = |b|$
Energy becomes REAL when $|g| > |b|$

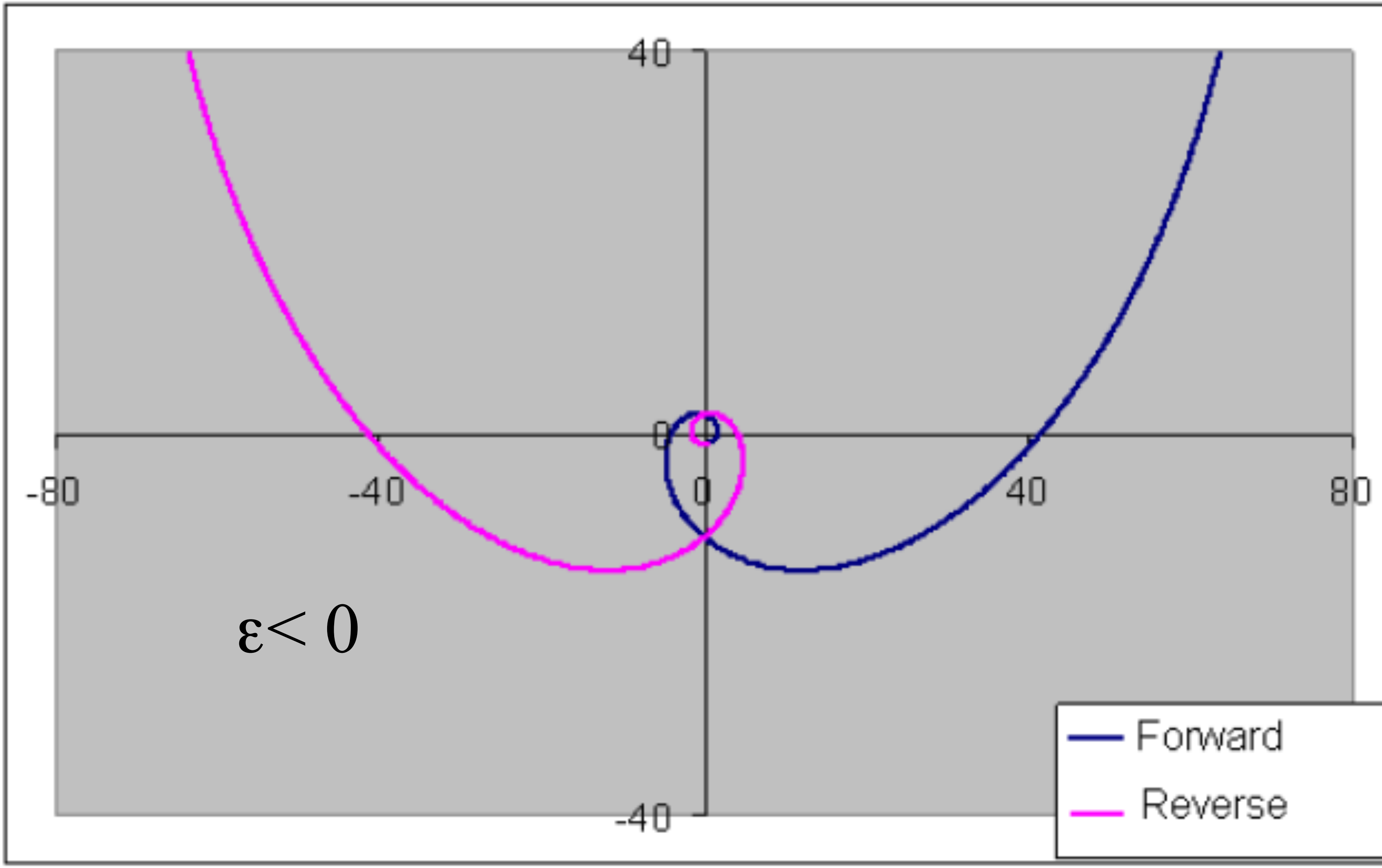
This system is in equilibrium for sufficiently large coupling!

PT phase transition at the classical level

ε = 1/2



Broken *PT* symmetry – orbit not closed



PT-symmetric systems lie between closed and open systems

Hermitian *H*

PT*-symmetric *H

Non-Hermitian *H*

Theoretical applications: renormalizing makes a Hamiltonian non-Hermitian, but still *PT* symmetric

- Lee model is unitary (there are no ghosts!)
- Pais-Uhlenbeck model (no ghosts!)
- Self-force on the electron (runaway modes)
- Double-scaling limit in QFT
- Stability of the Higgs vacuum
- Asymptotic behavior of the Painlevé transcendents
- Application to the Riemann hypothesis

…and many many many many more!

Experimental Studies of *PT* symmetry:

- *PT*-symmetric wave guides
- *PT*-symmetric lasers
- *PT*-symmetric electronic and mechanical systems
- Unidirectional transmission of light
- *PT*-symmetric atomic diffusion
- *PT*-symmetric superconducting wires
- *PT*-symmetric optical graphene and metamaterials

…and many many many many more!

PT–symmetric systems are being observed experimentally!

First observation of *PT* transition using optical wave guides:

“Observation of PT-symmetry breaking in complex optical potentials,” A. Guo, G. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. Siviloglou, and D. Christodoulides, *Physical Review Letters* **103**, 093902 (2009)

The *PT* adjoint and the *C* operator

nature physics

LETTERS

PUBLISHED ONLINE: 24 JANUARY 2010 | DOI: 10.1038/NPHYS1515

Observation of parity–time symmetry in optics

Christian E. Rüter[1], Konstantinos G. Makris[2], Ramy El-Ganainy[2], Demetrios N. Christodoulides[2], Mordechai Segev[3] and Detlef Kip[1]*

One of the fundamental axioms of quantum mechanics is associated with the Hermiticity of physical observables[1]. In the case of the Hamiltonian operator, this requirement not only implies real eigenenergies but also guarantees probability conservation. Interestingly, a wide class of non-Hermitian Hamiltonians can still show entirely real spectra. Among these are Hamiltonians respecting parity–time (*PT*) symmetry[2–7]. Even though the Hermiticity of quantum observables was never in doubt, such concepts have motivated discussions on several fronts in physics, including quantum field theories[8], non-Hermitian Anderson models[9] and open quantum systems[10,11], to mention a few. Although the impact of *PT* symmetry in these fields is still debated, it has been recently realized that optics can provide a fertile ground where *PT*-related notions can be implemented and experimentally investigated[12–15]. In this letter we report the first observation of the behaviour of a *PT* optical coupled system that judiciously involves a complex index potential. We observe both spontaneous *PT* symmetry breaking and power oscillations violating left–right symmetry. Our results may pave the way towards a new class of *PT*-synthetic materials with intriguing and unexpected properties that rely on non-reciprocal light propagation and tailored transverse energy flow.

($\varepsilon > \varepsilon_{th}$), the spectrum ceases to be real and starts to involve imaginary eigenvalues. This signifies the onset of a spontaneous *PT* symmetry-breaking, that is, a 'phase transition' from the exact to broken-*PT* phase[7,20].

In optics, several physical processes are known to obey equations that are formally equivalent to that of Schrödinger in quantum mechanics. Spatial diffraction and temporal dispersion are perhaps the most prominent examples. In this work we focus our attention on the spatial domain, for example optical beam propagation in *PT*-symmetric complex potentials. In fact, such *PT* 'optical potentials' can be realized through a judicious inclusion of index guiding and gain/loss regions[7,12–14]. Given that the complex refractive-index distribution $n(x) = n_R(x) + in_I(x)$ plays the role of an optical potential, we can then design a *PT*-symmetric system by satisfying the conditions $n_R(x) = n_R(-x)$ and $n_I(x) = -n_I(-x)$.

In other words, the refractive-index profile must be an even function of position x whereas the gain/loss distribution should be odd. Under these conditions, the electric-field envelope E of the optical beam is governed by the paraxial equation of diffraction[13]:

$$i\frac{\partial E}{\partial z} + \frac{1}{2k}\frac{\partial^2 E}{\partial x^2} + k_0[n_R(x) + in_I(x)]E = 0$$

PT-symmetric diffusion – Shanghai/Rutgers

PHYSICAL REVIEW A **81**, 042903 (2010)

Enhanced magnetic resonance signal of spin-polarized Rb atoms near surfaces of coated cells

K. F. Zhao,[1,*] M. Schaden,[2] and Z. Wu[2]

[1]*Institute of Modern Physics, Fudan University, Shanghai 200433, People's Republic of China*

[2]*Department of Physics, Rutgers University, Newark, New Jersey 07102, USA*

(Received 12 November 2009; published 21 April 2010)

We present a detailed experimental and theoretical study of edge enhancement in optically pumped Rb vapor in coated cylindrical pyrex glass cells. The Zeeman polarization of Rb atoms is produced and probed in the vicinity ($\sim 10^{-4}$ cm) of the cell surface by evanescent pump and probe beams. Spin-polarized Rb atoms diffuse throughout the cell in the presence of magnetic field gradients. In the present experiment the edge enhanced signal from the back surface of the cell is suppressed compared to that from the front surface, due to the fact that polarization is probed by the evanescent wave at the front surface only. The observed magnetic resonance line shape is reproduced quantitatively by a theoretical model and yields information about the dwell time and relaxation probability of Rb atoms on Pyrex glass surfaces coated with antirelaxation coatings.

DOI: 10.1103/PhysRevA.81.042903 PACS number(s): 34.35.+a, 75.40.Gb, 76.70.Hb, 87.57.nt

PT-symmetric optics – Caltech

SCIENCE VOL 333 5 AUGUST 2011

Nonreciprocal Light Propagation in a Silicon Photonic Circuit

Liang Feng,[1,2,4]*† Maurice Ayache,[3]* Jingqing Huang,[1,4]* Ye-Long Xu,[2] Ming-Hui Lu,[2] Yan-Feng Chen,[2]† Yeshaiahu Fainman,[3] Axel Scherer[1,4]†

Optical communications and computing require on-chip nonreciprocal light propagation to isolate and stabilize different chip-scale optical components. We have designed and fabricated a metallic-silicon waveguide system in which the optical potential is modulated along the length of the waveguide such that nonreciprocal light propagation is obtained on a silicon photonic chip. Nonreciprocal light transport and one-way photonic mode conversion are demonstrated at the wavelength of 1.55 micrometers in both simulations and experiments. Our system is compatible with conventional complementary metal-oxide-semiconductor processing, providing a way to chip-scale optical isolators for optical communications and computing.

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[1]Department of Electrical Engineering, California Institute of Technology, Pasadena, CA 91125, USA. [2]Nanjing National Laboratory of Microstructures, Nanjing University, Nanjing, Jiangsu 210093, China. [3]Department of Electrical and Computer Engineering, University of California, San Diego, La Jolla, CA 92093, USA. [4]Kavli Nanoscience Institute, California Institute of Technology, Pasadena, CA 91125, USA.

PT-symmetric superconducting wires – Indiana

PRL **99,** 167003 (2007) PHYSICAL REVIEW LETTERS week ending 19 OCTOBER 2007

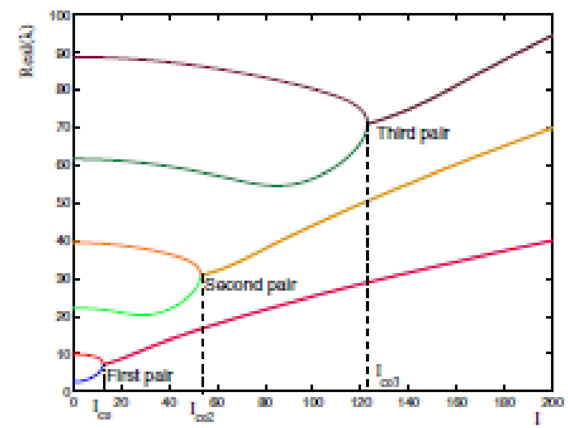
Bifurcation Diagram and Pattern Formation of Phase Slip Centers in Superconducting Wires Driven with Electric Currents

J. Rubinstein, P. Sternberg, and Q. Ma

Mathematics Department, Indiana University, Bloomington, Indiana 47405, USA

(Received 14 February 2007; published 18 October 2007)

We provide here new insights into the classical problem of a one-dimensional superconducting wire exposed to an applied electric current using the time-dependent Ginzburg-Landau model. The most striking feature of this system is the well-known appearance of oscillatory solutions exhibiting phase slip centers (PSC's) where the order parameter vanishes. Retaining temperature and applied current as parameters, we present a simple yet definitive explanation of the mechanism within this nonlinear model that leads to the PSC phenomenon and we establish where in parameter space these oscillatory solutions can be found. One of the most interesting features of the analysis is the evident collision of real eigenvalues of the associated *PT*-symmetric linearization, leading as it does to the emergence of complex elements of the spectrum.



PT-symmetric microwave cavities – Germany

PRL 108, 024101 (2012) PHYSICAL REVIEW LETTERS week ending 13 JANUARY 2012

$\mathcal{PT}$ Symmetry and Spontaneous Symmetry Breaking in a Microwave Billiard

S. Bittner,[1] B. Dietz,[1,*] U. Günther,[2] H. L. Harney,[3] M. Miski-Oglu,[1] A. Richter,[1,4,†] and F. Schäfer[1,5]

[1]*Institut für Kernphysik, Technische Universität Darmstadt, D-64289 Darmstadt, Germany*

[2]*Helmholtz-Zentrum Dresden-Rossendorf, Postfach 510119, D-01314 Dresden, Germany*

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[4]*ECT*, Villa Tambosi, I-38123 Villazzano (Trento), Italy*

[5]*LENS, University of Florence, I-50019 Sesto-Fiorentino (Firenze), Italy*

(Received 21 July 2011; published 10 January 2012)

We demonstrate the presence of parity-time ($\mathcal{PT}$) symmetry for the non-Hermitian two-state Hamiltonian of a dissipative microwave billiard in the vicinity of an exceptional point (EP). The shape of the billiard depends on two parameters. The Hamiltonian is determined from the measured resonance spectrum on a fine grid in the parameter plane. After applying a purely imaginary diagonal shift to the Hamiltonian, its eigenvalues are either real or complex conjugate on a curve, which passes through the EP. An appropriate basis choice reveals its $\mathcal{PT}$ symmetry. Spontaneous symmetry breaking occurs at the EP.

DOI: 10.1103/PhysRevLett.108.024101 PACS numbers: 05.45.Mt, 02.10.Yn, 11.30.Er

$\mathcal{PT}$-symmetric cavity lasers – Yale

PRL 106, 093902 (2011) PHYSICAL REVIEW LETTERS week ending 4 MARCH 2011

$\mathcal{PT}$-Symmetry Breaking and Laser-Absorber Modes in Optical Scattering Systems

Y. D. Chong,* Li Ge,† and A. Douglas Stone

Department of Applied Physics, Yale University, New Haven, Connecticut 06520, USA

(Received 30 August 2010; revised manuscript received 27 January 2011; published 2 March 2011)

Using a scattering matrix formalism, we derive the general scattering properties of optical structures that are symmetric under a combination of parity and time reversal ($\mathcal{PT}$). We demonstrate the existence of a transition between $\mathcal{PT}$-symmetric scattering eigenstates, which are norm preserving, and symmetry-broken pairs of eigenstates exhibiting net amplification and loss. The system proposed by Longhi [Phys. Rev. A **82**, 031801 (2010).], which can act simultaneously as a laser and coherent perfect absorber, occurs at discrete points in the broken-symmetry phase, when a pole and zero of the S matrix coincide.

DOI: 10.1103/PhysRevLett.106.093902 PACS numbers: 42.25.Bs, 42.25.Hz, 42.55.Ah

PT-symmetric photonic graphene – Israel

RAPID COMMUNICATIONS

PHYSICAL REVIEW A **84**, 021806(R) (2011)

$\mathcal{PT}$-symmetry in honeycomb photonic lattices

Alexander Szameit, Mikael C. Rechtsman, Omri Bahat-Treidel, and Mordechai Segev
Physics Department and Solid State Institute, Technion, 32000 Haifa, Israel
(Received 21 April 2011; published 19 August 2011)

We apply gain and loss to honeycomb photonic lattices and show that the dispersion relation is identical to tachyons—particles with imaginary mass that travel faster than the speed of light. This is accompanied by $\mathcal{PT}$-symmetry breaking in this structure. We further show that the $\mathcal{PT}$-symmetry can be restored by deforming the lattice.

DOI: 10.1103/PhysRevA.84.021806 PACS number(s): 42.25.−p, 42.82.Et

PT lasers – Vienna/Princeton/Yale/Zurich

PRL **108,** 173901 (2012) PHYSICAL REVIEW LETTERS week ending 27 APRIL 2012

Pump-Induced Exceptional Points in Lasers

M. Liertzer,[1,*] Li Ge,[2] A. Cerjan,[3] A. D. Stone,[3] H. E. Türeci,[2,4] and S. Rotter[1,†]

[1]*Institute for Theoretical Physics, Vienna University of Technology, A-1040 Vienna, Austria, EU*
[2]*Department of Electrical Engineering, Princeton University, Princeton, New Jersey 08544, USA*
[3]*Department of Applied Physics, Yale University, New Haven, Connecticut 06520, USA*
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(Received 2 September 2011; revised manuscript received 20 January 2012; published 24 April 2012)

We demonstrate that the above-threshold behavior of a laser can be strongly affected by exceptional points which are induced by pumping the laser nonuniformly. At these singularities, the eigenstates of the non-Hermitian operator which describes the lasing modes coalesce. In their vicinity, the laser may turn off even when the overall pump power deposited in the system is increased. Such signatures of a pump-induced exceptional point can be experimentally probed with coupled ridge or microdisk lasers.

Multiple *PT*-symmetric waveguides – Germany/Florida

ARTICLE

doi:10.1038/nature11298

Parity–time synthetic photonic lattices

Alois Regensburger[1,2], Christoph Bersch[1,2], Mohammad-Ali Miri[3], Georgy Onishchukov[2], Demetrios N. Christodoulides[3] & Ulf Peschel[1]

The development of new artificial structures and materials is today one of the major research challenges in optics. In most studies so far, the design of such structures has been based on the judicious manipulation of their refractive index properties. Recently, the prospect of simultaneously using gain and loss was suggested as a new way of achieving optical behaviour that is at present unattainable with standard arrangements. What facilitated these quests is the recently developed notion of 'parity–time symmetry' in optical systems, which allows a controlled interplay between gain and loss. Here we report the experimental observation of light transport in large-scale temporal lattices that are parity–time symmetric. In addition, we demonstrate that periodic structures respecting this symmetry can act as unidirectional invisible media when operated near their exceptional points. Our experimental results represent a step in the application of concepts from parity–time symmetry to a new generation of multifunctional optical devices and networks.

PT-symmetric superconducting wires – Argonne

PRL **109**, 150405 (2012) PHYSICAL REVIEW LETTERS week ending 12 OCTOBER 2012

Stimulation of the Fluctuation Superconductivity by $\mathcal{PT}$ Symmetry

N. M. Chtchelkatchev,[1,2] A. A. Golubov,[3] T. I. Baturina,[4,5] and V. M. Vinokur[4]

[1]*Institute for High Pressure Physics, Russian Academy of Sciences, Troitsk 142190, Moscow region, Russia*

[2]*Department of Theoretical Physics, Moscow Institute of Physics and Technology, 141700 Moscow, Russia*

[3]*Faculty of Science and Technology and MESA+ Institute of Nanotechnology, University of Twente, Enschede, The Netherlands*

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(Received 6 May 2012; published 9 October 2012)

We discuss fluctuations near the second-order phase transition where the free energy has an additional non-Hermitian term. The spectrum of the fluctuations changes when the odd-parity potential amplitude exceeds the critical value corresponding to the $\mathcal{PT}$-symmetry breakdown in the topological structure of the Hilbert space of the effective non-Hermitian Hamiltonian. We calculate the fluctuation contribution to the differential resistance of a superconducting weak link and find the manifestation of the $\mathcal{PT}$-symmetry breaking in its temperature evolution. We successfully validate our theory by carrying out measurements of far from equilibrium transport in mesoscale-patterned superconducting wires.

DOI: 10.1103/PhysRevLett.109.150405 PACS numbers: 11.30.Er, 03.65.Ge, 73.63.−b

PT-symmetric NMR – Beijing

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A

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Research

Cite this article: Zheng C, Hao L, Long GL. 2013 Observation of a fast evolution in a parity-time-symmetric system. Phil Trans R Soc A 371: 20120053.
http://dx.doi.org/10.1098/rsta.2012.0053

One contribution of 17 to a Theme Issue '$\mathcal{PT}$ quantum mechanics'.

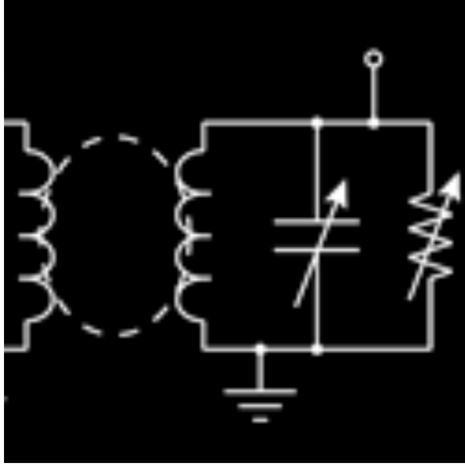
Observation of a fast evolution in a parity-time-symmetric system

Chao Zheng[1], Liang Hao[1] and Gui Lu Long[1,2]

[1]State Key Laboratory of Low-dimensional Quantum Physics and Department of Physics, Tsinghua University, Beijing 100084, People's Republic of China
[2]Tsinghua National Laboratory for Information Science and Technology, Beijing 100084, People's Republic of China

In parity-time-symmetric ($\mathcal{PT}$-symmetric) Hamiltonian theory, the optimal evolution time can be reduced drastically and can even be zero. In this article, we report our experimental simulation of the fast evolution of a $\mathcal{PT}$-symmetric Hamiltonian in a nuclear magnetic resonance quantum system. The experimental results demonstrate that the $\mathcal{PT}$-symmetric Hamiltonian system can indeed evolve much faster than the quantum system, and the evolution time can be arbitrarily close to zero.

APS: Spotlighting exceptional research



J. Schindler *et al.*, Phys. Rev. A (2011)

Experimental study of active *LRC* circuits with *PT* symmetries

Joseph Schindler, Ang Li, Mei C. Zheng, F. M. Ellis, and Tsampikos Kottos

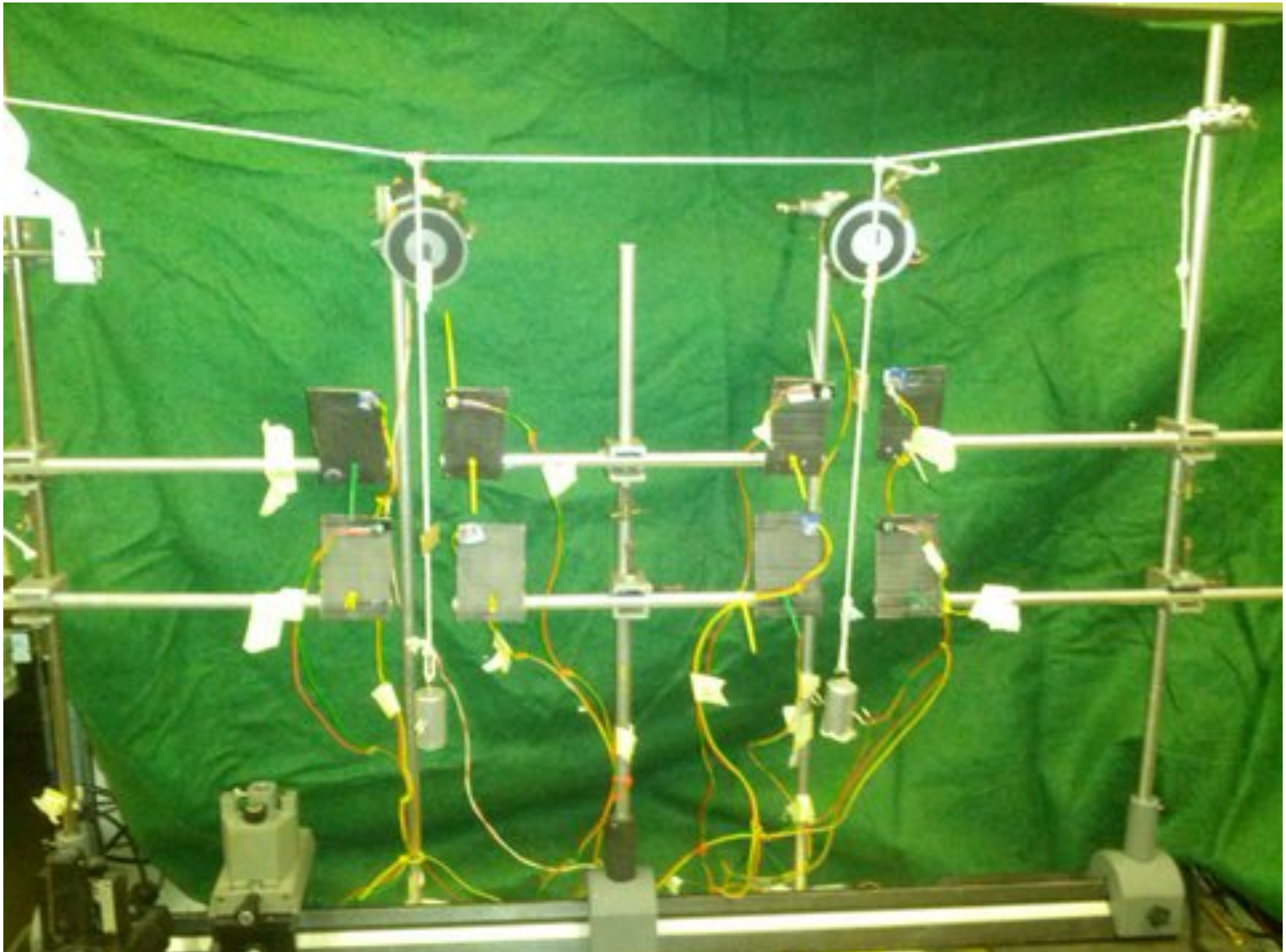
Phys. Rev. A **84**, 040101 (2011)

Published October 13, 2011

Everyone learns in a first course on quantum mechanics that the result of a measurement cannot be a complex number, so the quantum mechanical operator that corresponds to a measurement must be Hermitian. However, certain classes of complex Hamiltonians that are not Hermitian can still have real eigenvalues. The key property of these Hamiltonians is that they are parity-time (*PT*) symmetric, that is, they are invariant under a mirror reflection and complex conjugation (which is equivalent to time reversal).

Hamiltonians that have *PT* symmetry have been used to describe the depinning of vortex flux lines in type-II superconductors and optical effects that involve a complex index of refraction, but there has never been a simple physical system where the effects of *PT* symmetry can be clearly understood and explored. Now, Joseph Schindler and colleagues at Wesleyan University in Connecticut have devised a simple *LRC* electrical circuit that displays directly the effects of *PT* symmetry. The key components are a pair of coupled resonant circuits, one with active gain and the other with an equivalent amount of loss. Schindler *et al.* explore the eigenfrequencies of this system as a function of the "gain/loss" parameter that controls the degree of amplification and attenuation of the system. For a critical value of this parameter, the eigenfrequencies undergo a spontaneous phase transition from real to complex values, while the eigenstates coalesce and acquire a definite chirality (handedness). This simple electronic analog to a quantum Hamiltonian could be a useful reference point for studying more complex applications.
– *Gordon W. F. Drake*

“Observation of **PT** phase transition in a simple mechanical system,”
CMB, B. Berntson, D. Parker, E. Samuel, *American Journal of Physics* **81**, 173 (2013)



***PT*-symmetric system of coupled pendula**

$$\begin{aligned} x''(t) + ax'(t) + x(t) + \varepsilon y(t) &= 0 \\ y''(t) - ay'(t) + y(t) + \varepsilon x(t) &= 0 \end{aligned}$$

Loss and gain:

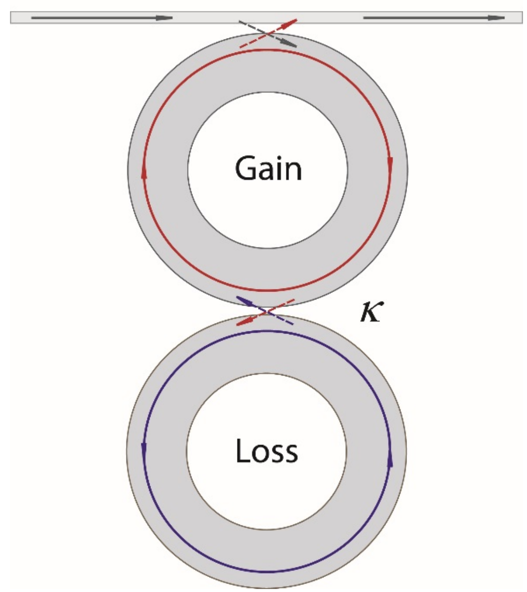
Remove energy from the *x* pendulum and transfer it to the *y* pendulum.

Recent fancy experiments involving whispering-galery microcavities

"Nonreciprocal light transmission in parity-time-symmetric whispering-gallery microcavities," B. Peng, S. K. Ozdemir, F. Lei, F. Monifi, M. Gianfreda, G. L. Long, S. Fan, F. Nori, CMB, L. Yang, *Nature Physics* **10**, 394 (2014)

"Twofold transition in **PT**-symmetric coupled oscillators," CMB, M. Gianfreda, B. Peng, S. K. Ozdemir, and L. Yang, *Physical Review A* **88**, 062111 (2013)

"Loss-induced suppression and revival of lasing," B. Peng, S.K. Ozdemir, S. Rotter, H. Yilmaz, M. Liertzer, CMB, F. Nori, L. Yang, *Science* **346**, 328 (2014)



PT-symmetric wireless power transfer – Stanford

LETTER

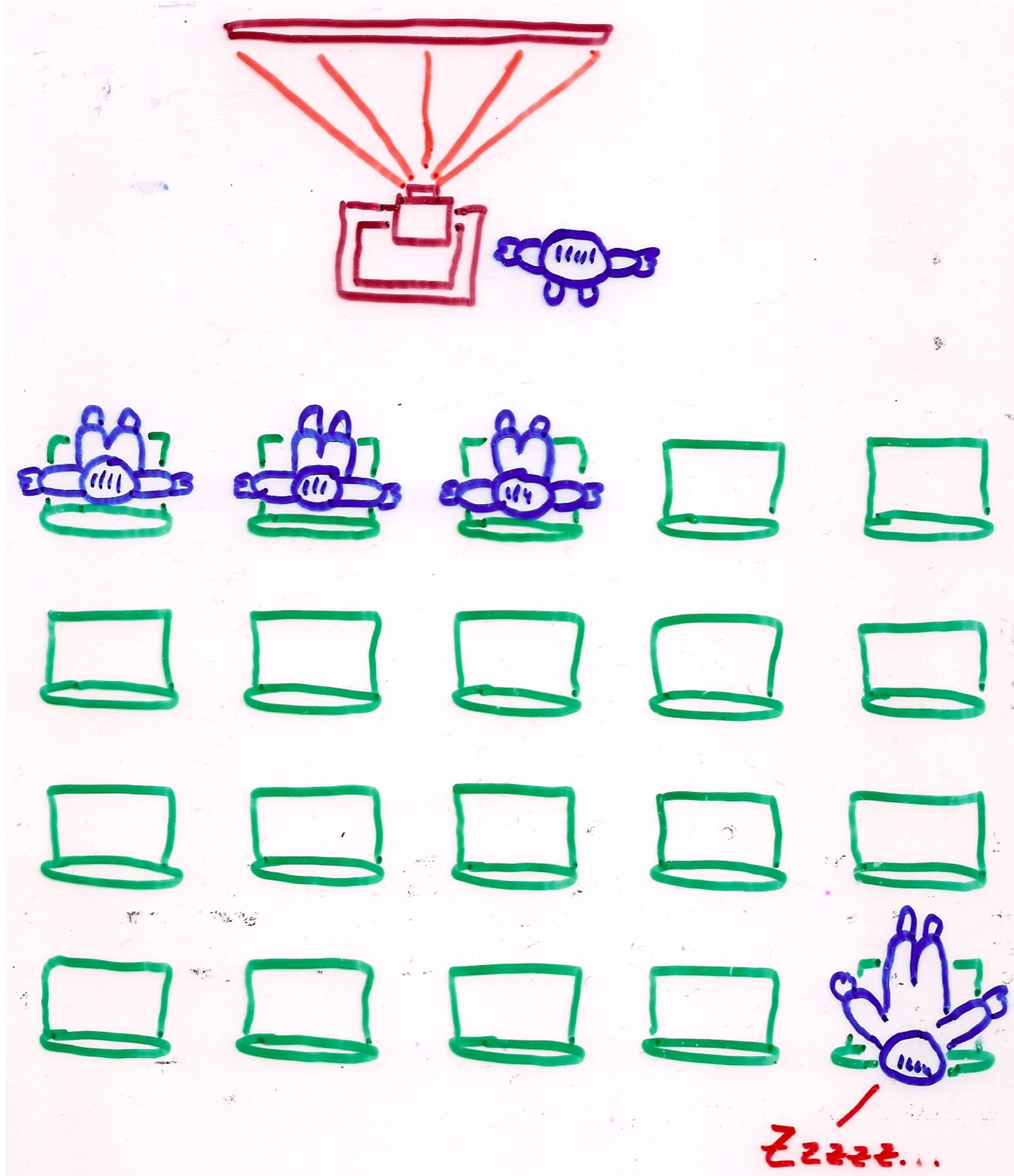
doi:10.1038/nature22404

Robust wireless power transfer using a nonlinear parity–time–symmetric circuit

Sid Assawaworrarit[1], Xiaofang Yu[1] & Shanhui Fan[1]

Considerable progress in wireless power transfer has been made in the realm of non-radiative transfer, which employs magnetic-field coupling in the near field[1–4]. A combination of circuit resonance and impedance transformation is often used to help to achieve efficient transfer of power over a predetermined distance of about the size of the resonators[3,4]. The development of non-radiative wireless power transfer has paved the way towards real-world applications such as wireless powering of implantable medical devices and wireless charging of stationary electric vehicles[1,2,5–8]. However, it remains a fundamental challenge to create a wireless power transfer system in which the transfer efficiency is robust against the variation of operating conditions. Here we propose theoretically and demonstrate experimentally that a parity–time-symmetric circuit incorporating a nonlinear gain saturation element provides robust wireless power transfer. Our results show that the transfer efficiency remains near unity over a distance variation of approximately one metre, without the need for any tuning. This is in contrast with conventional methods where high transfer efficiency can only be maintained by constantly tuning the frequency or the internal coupling parameters as the transfer distance or the relative orientation of the source and receiver units is varied. The use of a nonlinear parity–time-symmetric circuit should enable robust wireless power transfer to moving devices or vehicles[9,10].

Overview of this course:



Theoretical examples:
Lee model

$$V \to N + \theta, \qquad N + \theta \to V.$$

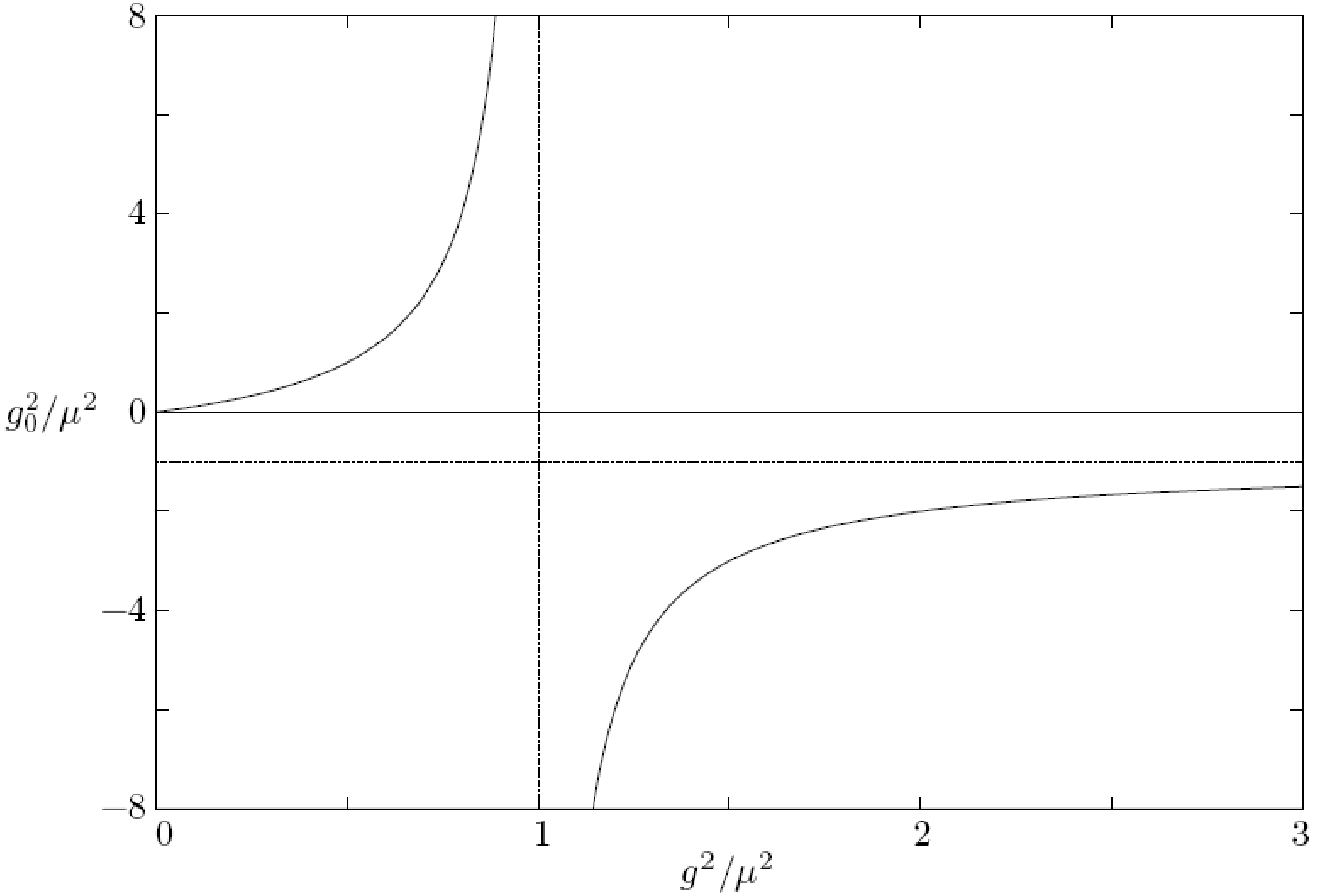
$$H = H_0 + g_0 H_1,$$

$$\begin{aligned} H_0 &= m_{V_0} V^\dagger V + m_N N^\dagger N + m_\theta a^\dagger a, \\ H_1 &= V^\dagger N a + a^\dagger N^\dagger V. \end{aligned}$$

T. D. Lee, Phys. Rev. **95**, 1329 (1954)

G. Källén and W. Pauli, Dan. Mat. Fys. Medd. **30**, No. 7 (1955)

Problem with the Lee model



$$g_0^2 = g^2 / \left(1 - g^2/\mu^2\right)$$

"A non-Hermitian Hamiltonian is unacceptable partly because it may lead to complex energy eigenvalues, but chiefly because it implies a non-unitary S matrix, which fails to conserve probability and makes a hash of the physical interpretation."

G. Barton, *Introduction to Advanced Field Theory* (John Wiley & Sons, New York, 1963)

Renormalization creates instability.
This is a *really* hard problem. Pauli, Heisenberg, Wick, Sudarshan, … worked on it, but no cigar.

GHOSTBUSTING: Reviving quantum theories that were thought to be dead

"Ghost busting: ***PT***-symmetric interpretation of the Lee model,"
CMB, S. Brandt, J.-H. Chen, and Q. Wang, *Phys. Rev. D* **71**, 025014 (2005)

PT symmetry and instabilities of nonlinear differential equations

Painlevé transcendents have fundamental instabilities that can be tamed and understood quantitatively by using *PT*-symmetric quantum theory

"Nonlinear eigenvalue problems,"
CMB, A. Fring, and J. Komijani,
Journal of Physics A: Mathematical and Theoretical **47**, 235204 (2014)

"**PT**-symmetric Hamiltonians and the Painlevé transcendents,"
CMB and J. Komijani,
Journal of Physics A: Mathematical and Theoretical **48**, 475202 (2015)

"Nonlinear eigenvalue problems"
CMB, J. Komijani, and Q. Wang,
In *Resurgence, Physics and Numbers*, ed. by F. Fauvet, D. Manchon, S. Marmi, and D. Sauzin
CRM (Centro di Ricerca Matematica) Series, Ennio De Giorgi **20**, 67-89 (2017)

Asymptotics beyond all orders

Leading asymptotic behavior of solutions to

$$-\psi''(x) + V(x)\psi(x) = E\psi(x)$$

for large positive *x*:

$$\psi(x) \sim C[V(x) - E]^{-1/4} \exp\left[\int^x ds \sqrt{V(s) - E}\right] \quad (x \to \infty)$$

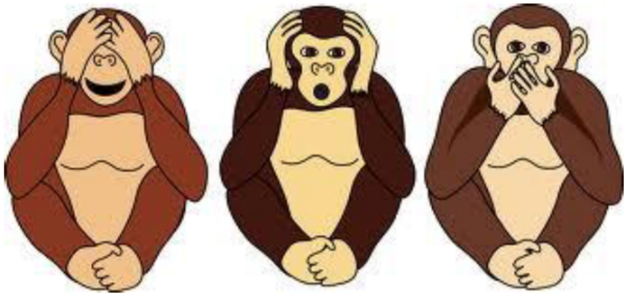
NOTE: There is only *ONE* arbitrary constant.

Second arbitrary constant is invisible with Poincaré asymptotics because it is contained in the *subdominant* solution:

$$\psi(x) \sim D[V(x) - E]^{-1/4} \exp\left[-\int^x ds \sqrt{V(s) - E}\right] \quad (x \to \infty)$$

***Physical* solution is *Unstable* under small changes in *E*.**

Eigenfunctions: 3 characteristic properties



(1) Oscillatory in *classically allowed* region (*n*th eigenfunction has *n* nodes)

(2) Monotone decay in *classically forbidden* region

(3) Transition at the boundary (*turning point*)

Toy nonlinear eigenvalue problem

$$y'(x) = \cos[\pi x y(x)], \quad y(0) = a$$

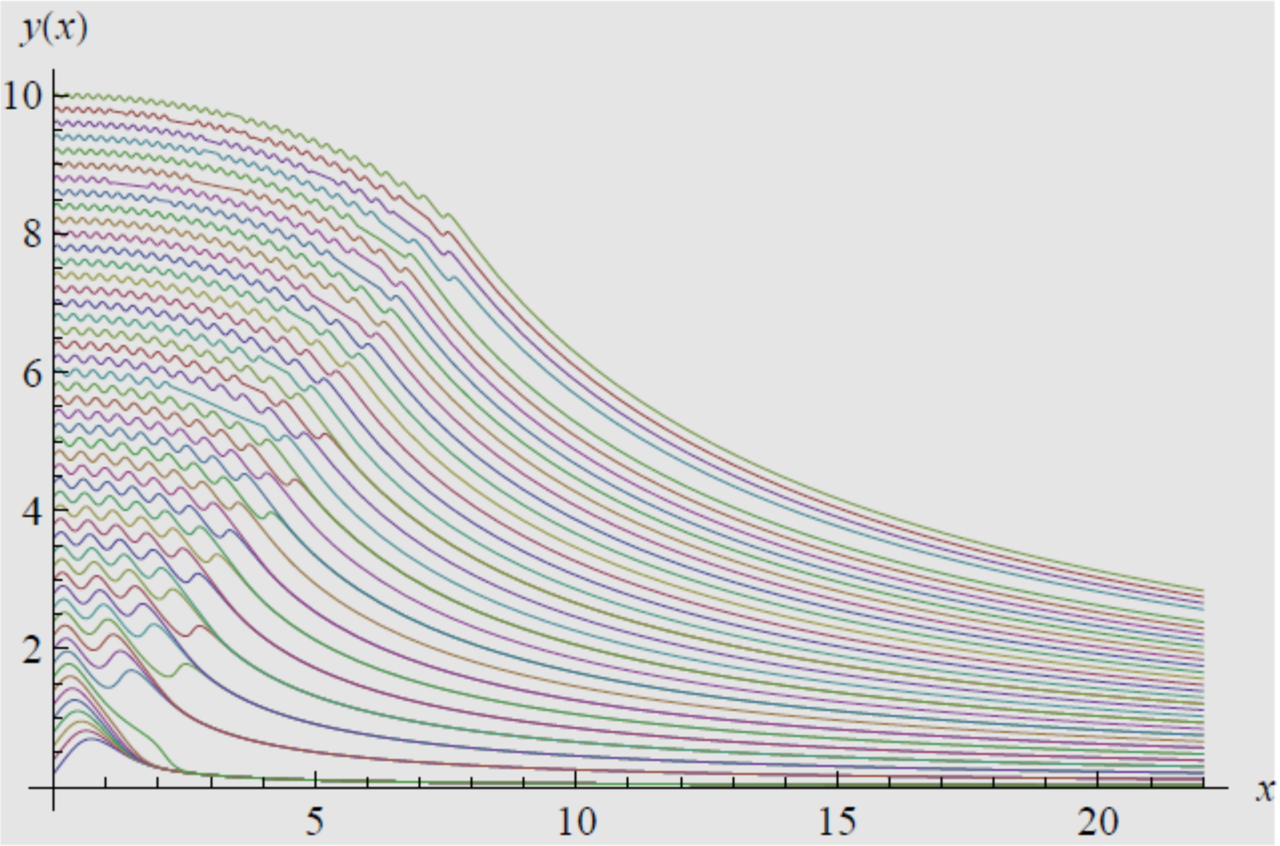
Some references:

[1] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw Hill, New York, 1978), chap. 4.

[2] C. M. Bender, D. W. Hook, P. N. Meisinger, and Q. Wang, Phys. Rev. Lett. **104**, 061601 (2010).

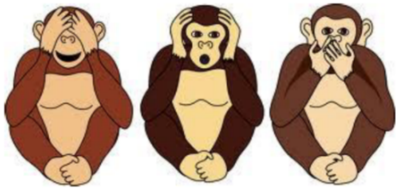
[3] C. M. Bender, D. W. Hook, P. N. Meisinger, and Q. Wang, Ann. Phys. **325**, 2332-2362 (2010).

[4] J. Gair, N. Yunes, and C. M. Bender, J. Math. Phys. **53**, 032503 (2012).



Solutions for 50 initial conditions

Note: (1) oscillation (2) monotone decay (3) transition

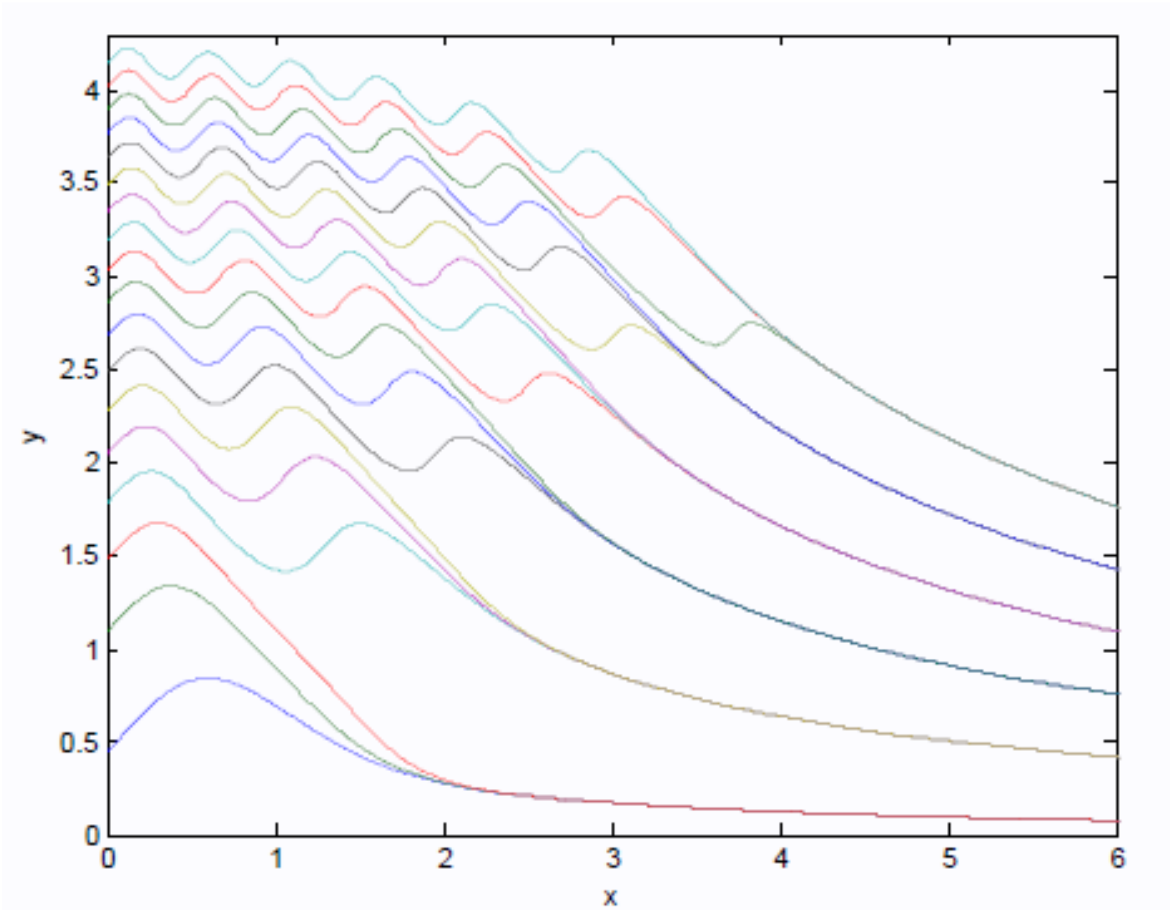


Asymptotic behavior for large *x*

Solution behaves like: $$y(x) \sim \frac{m + 1/2}{x}$$

***m* = 0, 1, 2, 3, ... is an integer**

There's a *big* problem here...



$m = 10$
$m = 8$
$m = 6$
$m = 4$
$m = 2$
$m = 0$

Where are the odd-*m* solutions?!?

Furthermore, no arbitrary constant appears in the asymptotic behavior!!

Where is the arbitrary constant?!?

Is it in higher order?

Higher-order asymptotic behavior for large *x* still contains <u>no arbitrary constant</u>!

$$y(x) \sim \frac{m+1/2}{x} + \sum_{k=1}^{\infty} \frac{c_k}{x^{2k+1}} \quad (x \to \infty)$$

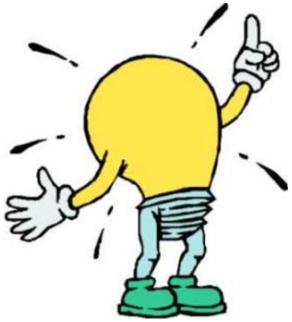
$$\begin{aligned}
c_1 &= \frac{(-1)^m}{\pi}(m+1/2), \\
c_2 &= \frac{3}{\pi^2}(m+1/2), \\
c_3 &= (-1)^m \left[\frac{(m+1/2)^3}{6\pi} + \frac{15(m+1/2)}{\pi^3}\right], \\
c_4 &= \frac{8(m+1/2)^3}{3\pi^2} + \frac{105(m+1/2)}{\pi^4}, \\
c_5 &= (-1)^m \left[\frac{3(m+1/2)^5}{40\pi} + \frac{36(m+1/2)^3}{\pi^3} + \frac{945(m+1/2)}{\pi^5}\right], \\
c_6 &= \frac{38(m+1/2)^5}{15\pi^2} + \frac{498(m+1/2)^3}{\pi^4} + \frac{10395(m+1/2)}{\pi^6}.
\end{aligned}$$

Asymptotics beyond all orders

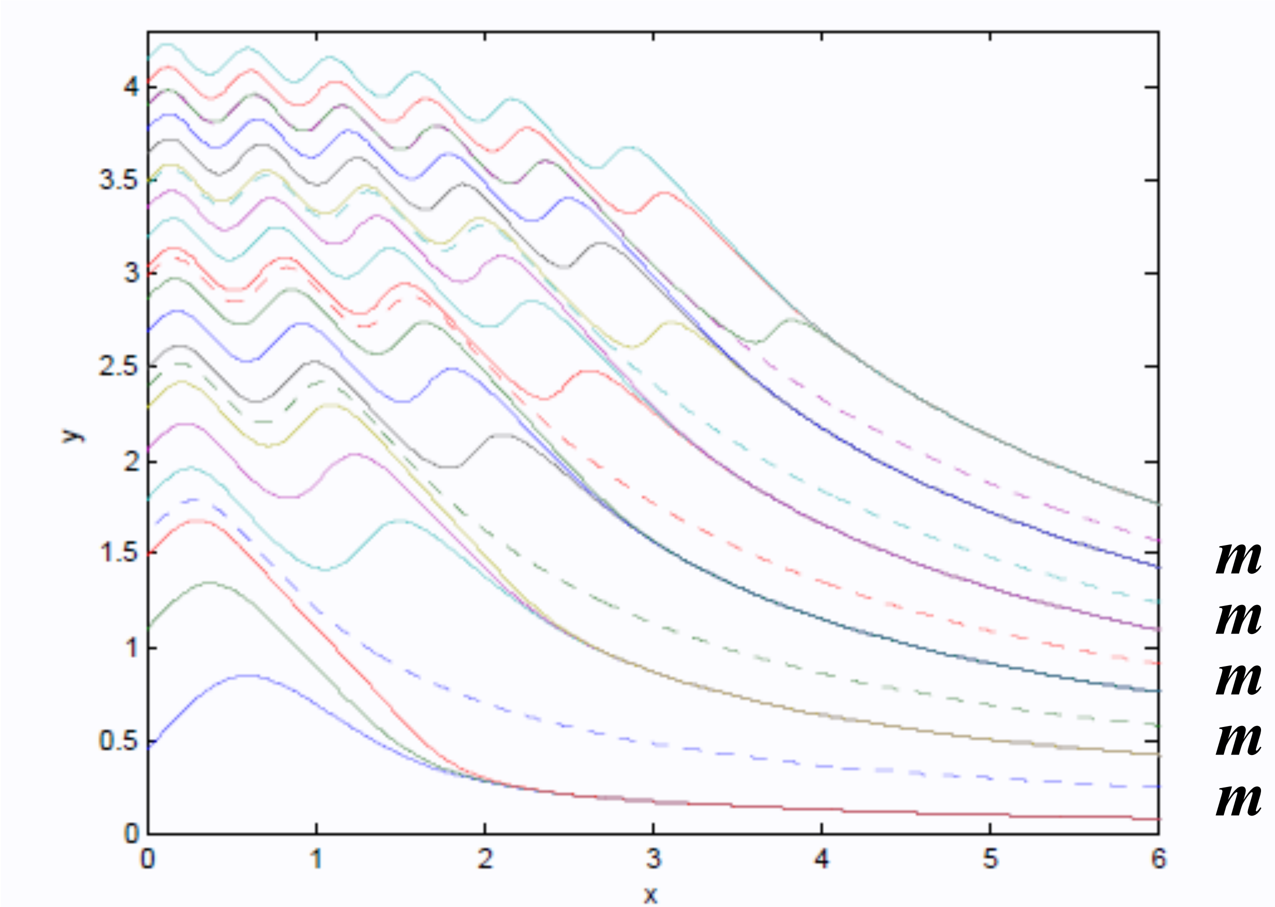
Difference of two solutions in one bundle: $Y(x) \equiv y_1(x) - y_2(x)$

$$\begin{aligned} Y'(x) &= \cos[\pi x y_1(x)] - \cos[\pi x y_2(x)] \\ &= -2\sin\left[\tfrac{1}{2}\pi x y_1(x) + \tfrac{1}{2}\pi x y_2(x)\right]\sin\left[\tfrac{1}{2}\pi x y_1(x) - \tfrac{1}{2}\pi x y_2(x)\right] \\ &\sim -2\sin\left[\pi\left(m+\tfrac{1}{2}\right)\right]\sin\left[\tfrac{1}{2}\pi x Y(x)\right] \quad (x\to\infty) \\ &\sim -(-1)^m \pi x Y(x) \quad (x\to\infty). \end{aligned}$$

$$Y(x) \sim K\exp\left[-(-1)^m \pi x^2\right] \quad (x\to\infty)$$



Aha! *K* is the invisible arbitrary constant!
Odd-*m* solutions are *unstable*;
even-*m* solutions are *stable*.



$$y(0) = a \in \{1.6026, 2.3884, 2.9767, 3.4675, 3.8975, 4.2847, \ldots\}$$

Eigenvalues **correspond to odd-*m* initial values.**
Eigenfunctions **are (*unstable*) *separatrices*, which begin at eigenvalues.**

We calculated up to *m*=500,001

Let $m = 2n - 1$

For large *n* the *n*th eigenvalue grows like the *square root* of *n* times a constant *A*, and we used Richardson extrapolation to show that

A= 1.7817974363...

and then we guessed *A*.

Result:

$$a_n \sim A\sqrt{n} \quad (n \to \infty)$$

$$A = 2^{5/6}$$

This is a rather nontrivial problem...

Analytic calculation of the constant A

Construct moments of $z(t)$:

$$A_{n,k}(t) \equiv \int_0^t ds \cos[n\lambda s z(s)] \frac{s^{k+1}}{[z(s)]^k}$$

Moments are associated with a semi-infinite linear one-dimensional random walk in which random walkers become static as they reach n=1

$$2\alpha_{1,k} + \alpha_{2,k-1} = 0, \qquad 2\alpha_{n,k} + \alpha_{n-1,k-1} + \alpha_{n+1,k-1} = 0 \quad (n \geq 3).$$

$$2\alpha_{2,k} + \alpha_{3,k-1} = 0,$$

Solve the random walk problem exactly and get $A = 2^{5/6}$

CMB, A. Fring, and J. Komijani
J. Phys. A: Math. Theor. **47**, 235204 (2014)
[arXiv: math-ph/1401.6161]

Possible connection with the *power series constant P*???

(Remember the numerical constant $A = 1.7818$)

W. K. Hayman, *Research Problems in Function theory*
[Athlone Press (University of London), London, 1967]

J. Clunie and P. Erdös, Proc. Roy. Irish Acad. **65**, 113 (1967).
J. D. Buckholtz, Michigan Math. J. **15**, 481 (1968).

$$1 \le P \le 2$$

$$\sqrt{2} \le P \le 2$$

$$1.7 \le P \le 12^{1/4}$$

$$1.7818 \le P \le 1.82$$

Three ***nontrivial*** second-order nonlinear eigenvalue problems

separatrix

Painlevé I

$$\frac{d^2y}{dt^2} = 6y^2 + t$$

Painlevé II

$$\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$$

Painlevé III

$$ty\frac{d^2y}{dt^2} = t\left(\frac{dy}{dt}\right)^2 - y\frac{dy}{dt} + \delta t + \beta y + \alpha y^3 + \gamma t y^4$$

Painlevé IV

$$y\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{dy}{dt}\right)^2 + \beta + 2(t^2 - \alpha)y^2 + 4ty^3 + \frac{3}{2}y^4$$

Painlevé V

$$\frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}$$

Painlevé VI

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

(1) First Painlevé transcendent

$$y''(t) = 6[y(t)]^2 + t, \qquad y(0) = b,\ y'(0) = c$$

Solution *y*(*x*) must *choose* between two possible asymptotic behaviors as *x* gets large and negative:

$$+\sqrt{-t/6} \text{ or } -\sqrt{-t/6}.$$

Example of a *difficult* choice ...

Two possible asymptotic behaviors

Lower square-root branch is *stable*:

$$y(x) \sim -\sqrt{-x} + c(-x)^{-1/8} \cos\left[\tfrac{4}{5}\sqrt{2}(-x)^{5/4} + d\right] \quad (x \to -\infty)$$

Upper square-root branch is *unstable*:

$$y(x) \sim \sqrt{-x} + c_{\pm}(-x)^{-1/8} \exp\left[\pm\tfrac{4}{5}\sqrt{2}(-x)^{5/4}\right] \quad (x \to -\infty)$$

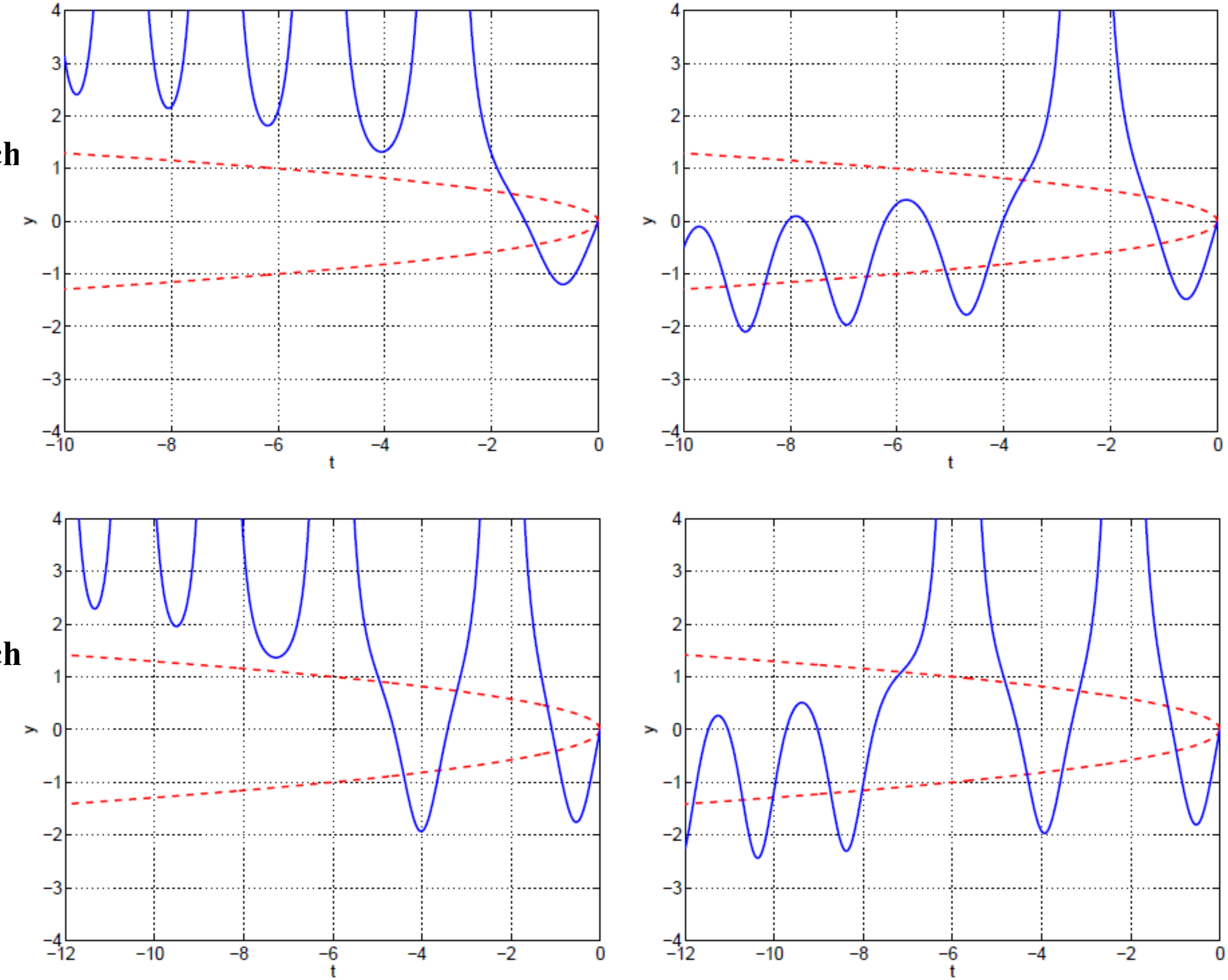
Two possible kinds of solutions (NOT eigenfunctions):

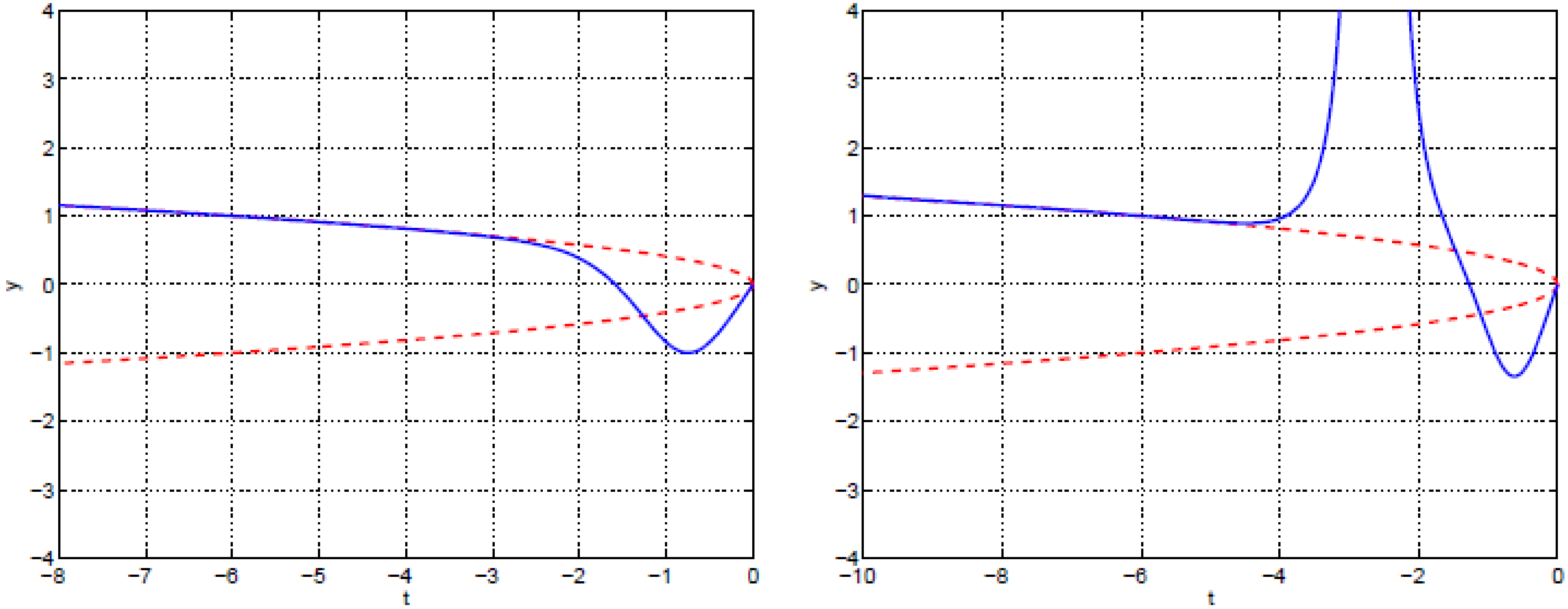
Unstable branch

Stable branch

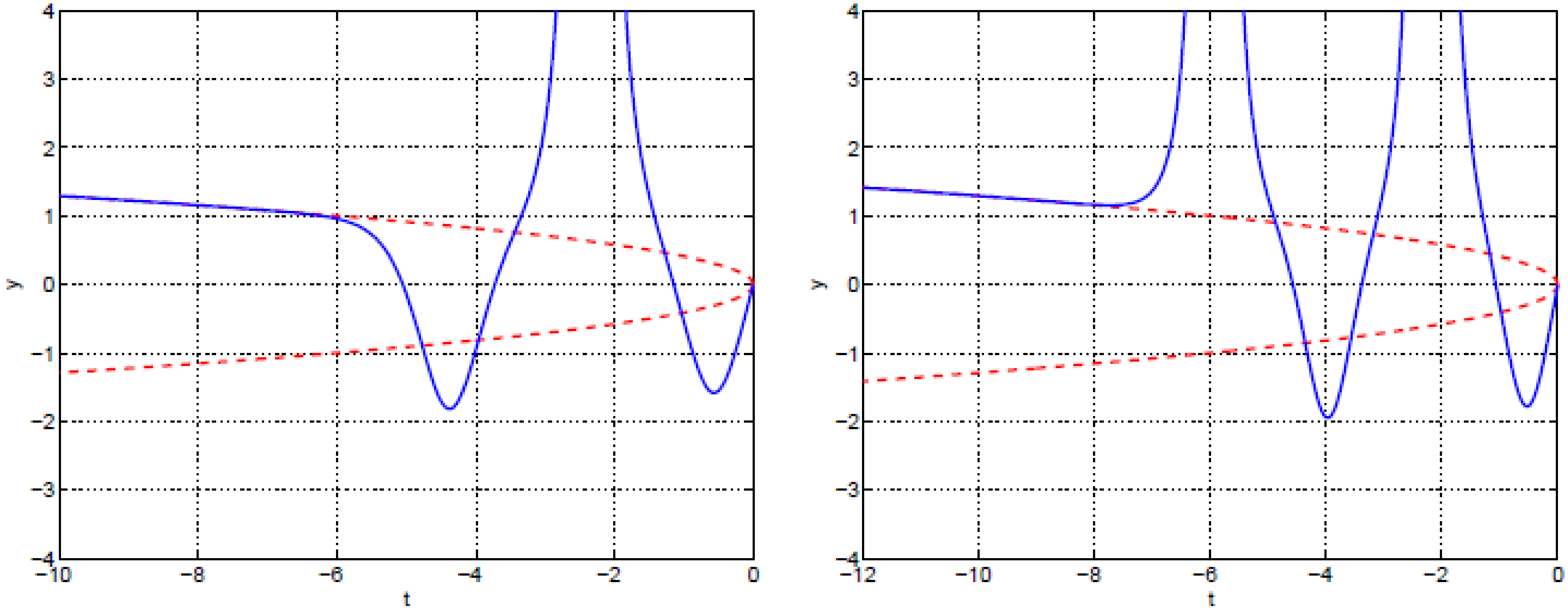
Unstable branch

Stable branch

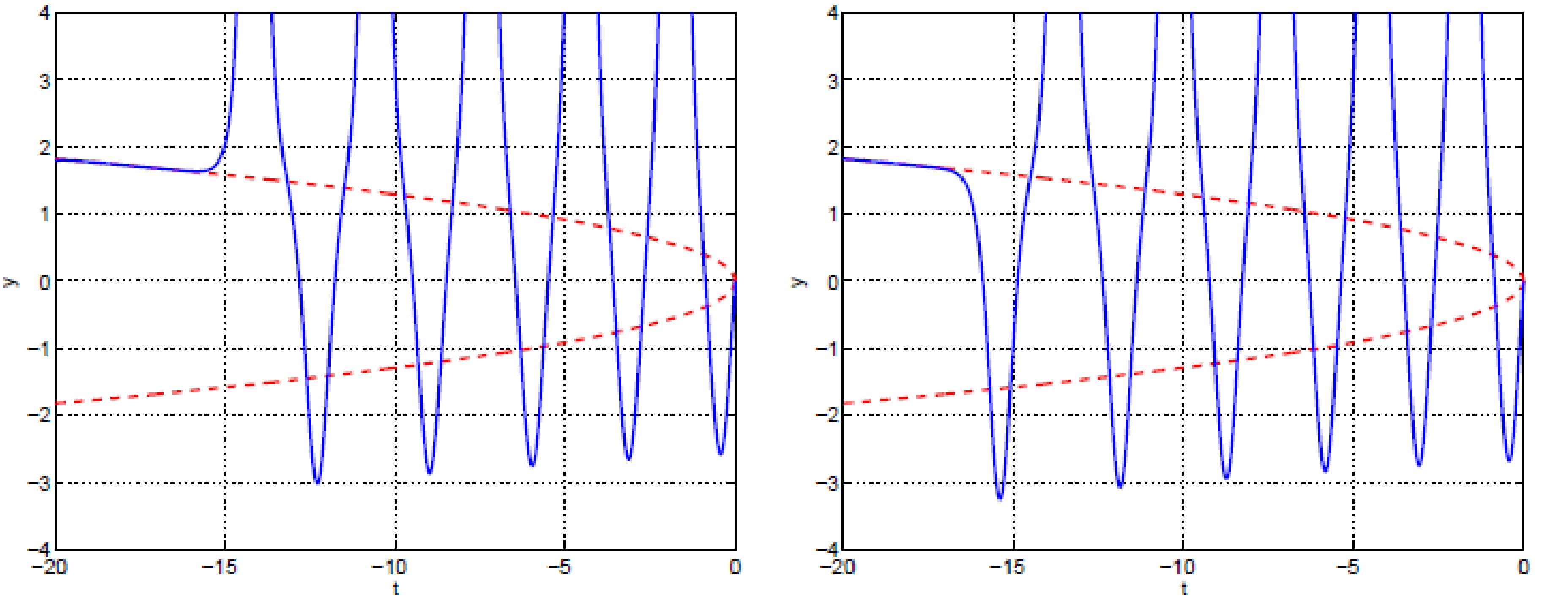




First two separatrix solutions (eigenfunctions) of Painlevé I with initial condition $y(0) = 0$. Left panel: $y'(0) = b_1 = 1.851854034$; right panel: $y'(0) = b_2 = 3.004031103$. The dashed curves are $y = \pm\sqrt{-t/6}$.



Third and fourth eigenfunctions of Painlevé I with initial condition $y(0) = 0$. Left panel: $y'(0) = b_3 = 3.905175320$; right panel: $y'(0) = b_4 = 4.683412410$.



Tenth and eleventh eigenfunctions of Painlevé I with initial condition $y(0) = 0$. Left panel: $y'(0) = b_{10} = 8.244932302$; right panel: $y'(0) = b_{11} = 8.738330156$. Note that as n increases, the eigenfunctions pass through more and more double poles before exhibiting a turning-point-like transition and approaching the limiting curve $+\sqrt{-t/6}$ exponentially rapidly. This behavior is analogous to that of the eigenfunctions of a time-independent Schrödinger equation for a particle in a potential well; the higher-energy eigenfunctions exhibit more and more oscillations in the classically allowed region before entering the classically forbidden region, where they decay exponentially.

For large *n*:

$$y'_n(0) = b_n \sim B_{\text{I}} n^{3/5}$$

Fifth-order Richardson applied to first 11 eigenvalues:

$$B_{\text{I}} = 2.09214674\underline{4}$$

Can you guess this number?

$$y_n(0) = c_n \sim C_{\text{I}} n^{2/5}$$

Fourth-order Richardson applied to first 15 eigenvalues:

$$C_{\text{I}} = -1.030484\underline{4}$$

Can you guess this number?

$$B_{\mathrm{I}} = 2\left[\sqrt{3\pi}\,\Gamma\left(\tfrac{11}{6}\right)/\Gamma\left(\tfrac{1}{3}\right)\right]^{3/5}$$

$$C_{\mathrm{I}} = -\left[\sqrt{3\pi}\,\Gamma\left(\tfrac{11}{6}\right)/\Gamma\left(\tfrac{1}{3}\right)\right]^{2/5}$$

Instability of Painlevé I explained from large eigenvalues of

cubic PT-symmetric Hamiltonian

$$H = p^2 + ix^3$$

Painlevé I corresponds to ε = 1

(Do you remember the cubic *PT*-symmetric Hamiltonian?!)

Instability of Painlevé II explained from large eigenvalues of

quartic PT-symmetric Hamiltonian

$$H = p^2 - x^4$$

Painlevé II corresponds to ε = 2

(Do you remember the quartic upside-down *PT*-symmetric Hamiltonian?!)

Instability of Painlevé IV explained in terms of the

sextic PT-symmetric Hamiltonian

$$H = p^2 + x^6$$

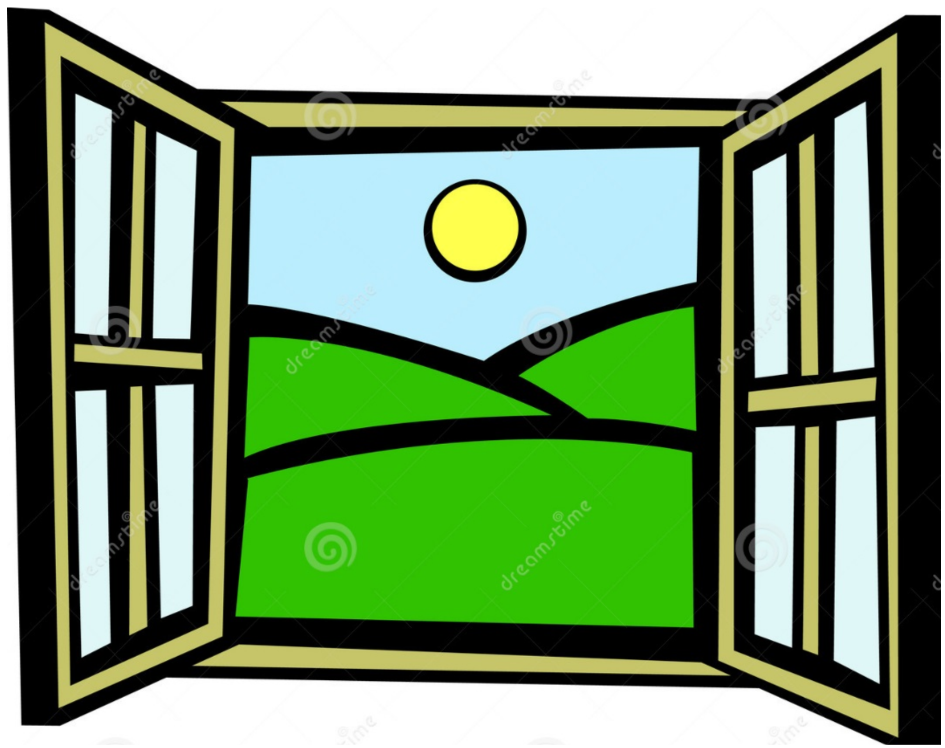
Painlevé IV corresponds to ε = 4

(Do you remember the sextic *PT*-symmetric Hamiltonian?!)

In general, this analysis works for huge classes of equations beyond Painlevé. For example:

$$y''(x) = \frac{2M+2}{(M-1)^2}[y(x)]^M + x[y(x)]^N$$

This is a new area of *nonlinear semiclassical* asymptotic analysis!



I hope you enjoyed this course!