Day #2 of a boring lecture series

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Summing divergent series

- Warning: addition is not infinitely commutative
- Warning: addition is not infinitely associative
- Euler summation
- Borel summation
- Generic summation methods
- Zeta summation
- Continued functions

Some practice series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots$$

$$1 + 0 - 1 + 1 - 0 + 1 + \cdots$$

$$1 + 1 + 1 + 1 + 1 + 1 + \cdots$$

Can we sum these series??!

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... = \infty$$

$$\zeta(0) = 1 + 1 + 1 + 1 + 1... = -\frac{1}{2}$$

$$\zeta(-1) = 1 + 2 + 3 + 4 + 5 + 6... = -\frac{1}{12}$$

This seems to require complex analysis…

Complex plane

Zeta summation

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \oint_C \mathrm{d}t \frac{t^{s-1}}{\mathrm{e}^{-t} - 1}$$

C **is a *Hankel contour* that encircles the negative-real *t* axis in the positive direction.**

For s = 0 and s = −1:

$$\zeta(0) = -\frac{1}{2}$$

$$\zeta(-1) = -\frac{1}{12}$$

Application of Euler summation!

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

converges for $\mathrm{Re}(s) > 0$ but diverges for $\mathrm{Re}(s) \leq 0$

$$\zeta(0) = -1 + 1 - 1 + 1 - 1 + 1 \cdots$$

$$f(t) = -\frac{1}{1+t}$$

$$\zeta(0) = f(1) = -\frac{1}{2}$$

For s = −1

We get the series: $$-\frac{1}{3}+\frac{2}{3}-\frac{3}{3}+\frac{4}{3}-\cdots$$

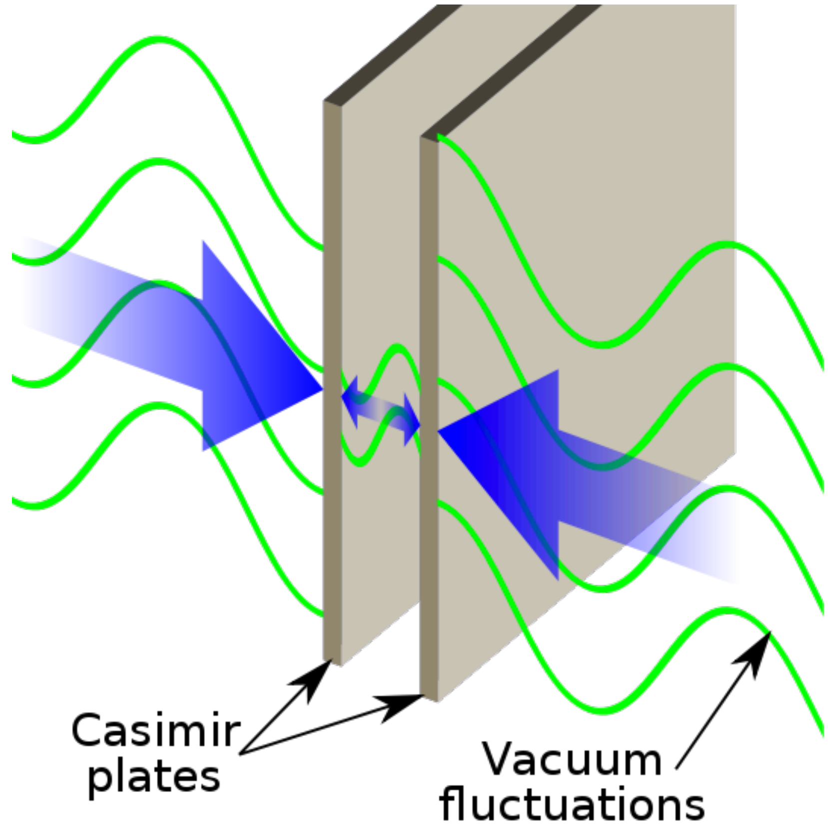
$$f(t) = -\frac{t}{3(1+t)^2}$$

$$\boxed{\zeta(-1) = f(1) = -\frac{1}{12}}$$

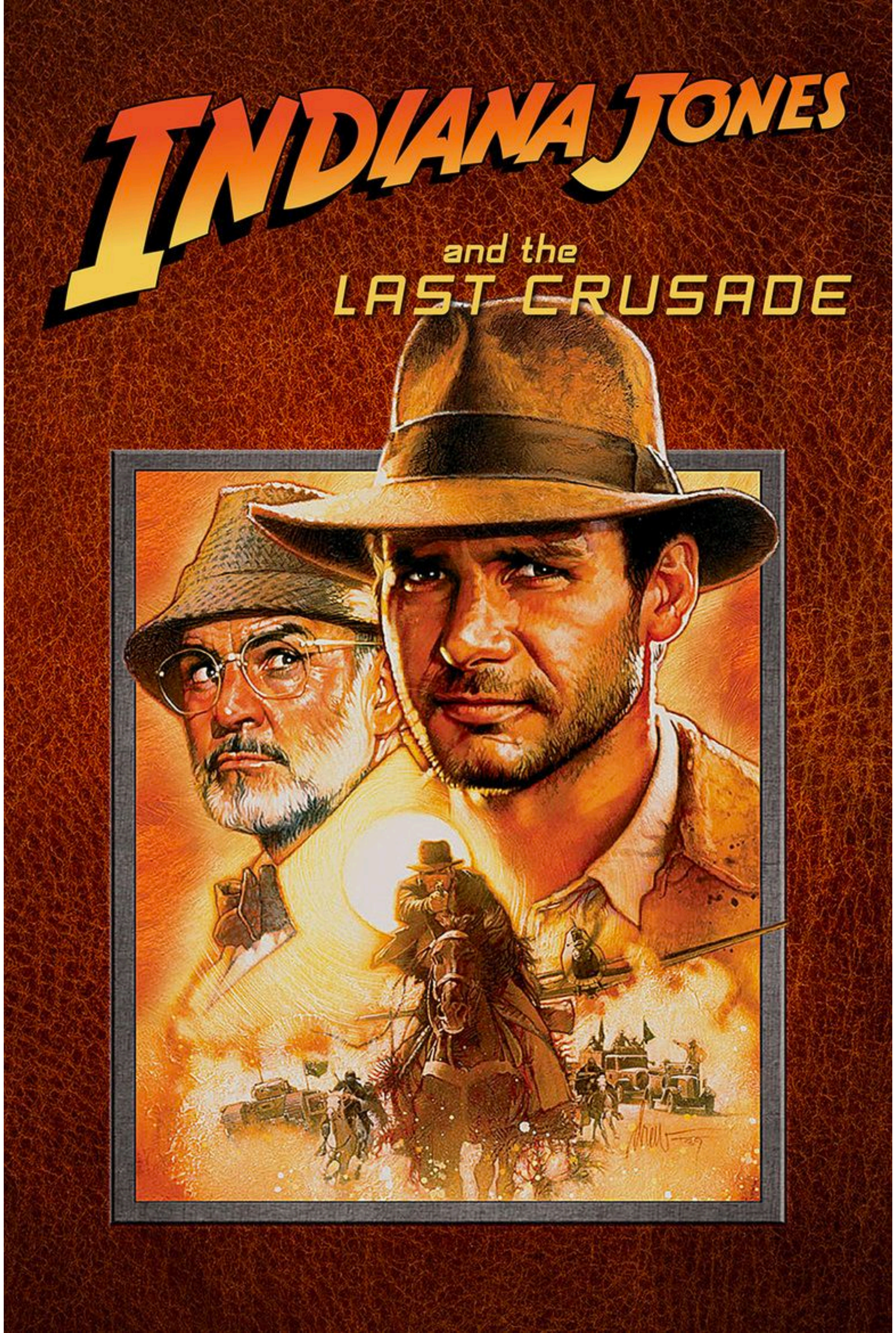
No complex numbers appear explicitly!!!

Application in physics

Hendrik Casimir
1909-2000



A remarkable idea…
the *Casimir force*
is a sum over modes



Dramatic example of the Casimir force…


TIBER

BOOM!

Euler summaton for quantum mechanics

Reconstructing a Hamiltonian from its eigenvalues and eigenstates:

$$\hat{H} = \sum_n E_n |\phi_n\rangle\langle\phi_n|$$

Completeness:

$$\delta(x-y) = \sum_n \langle x|n\rangle\langle n|y\rangle$$

These are divergent series!

Example: Square well on [0,π]

$$\hat{H} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

$$\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx) \quad \text{and} \quad E_n = \tfrac{1}{2}n^2$$

$$H(x,y) = \frac{1}{\pi}\sum_{n=1}^{\infty} n^2 \sin(n\pi x)\sin(n\pi y) = -\frac{1}{2}\frac{\partial^2}{\partial y^2}\delta(x-y)$$

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\sin(nx)\sin(ny) = \delta(x-y)$$

$$K(x,y,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} t^n \sin(nx)\sin(ny)$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty} t^n \{\cos[n(x-y)] - \cos[n(x+y)]\}$$

$$= \quad D(x-y,t) - D(x+y,t)$$

$$D(z,t) = \frac{1 - t\cos z}{\pi(1 - 2t\cos z + t^2)}$$

As t approaches 1 you get a delta function $\delta(x - y)$

We discussed:

***Borel* summation**

and

***Generic* summation**

Crucial problem:

What if we don't know all the terms in the series??

This is a nontrivial problem...

Example of a *nontrivial* problem

...and here is the solution!

Example: Continued exponentials

Rewrite the divergent series $\sum_{n=0}^{\infty} c_n z^n$

as the continued exponential $a_0 e^{a_1 z e^{a_2 z \cdots}}$

$$c_0 = a_0,$$

$$c_1 = a_1 a_0,$$

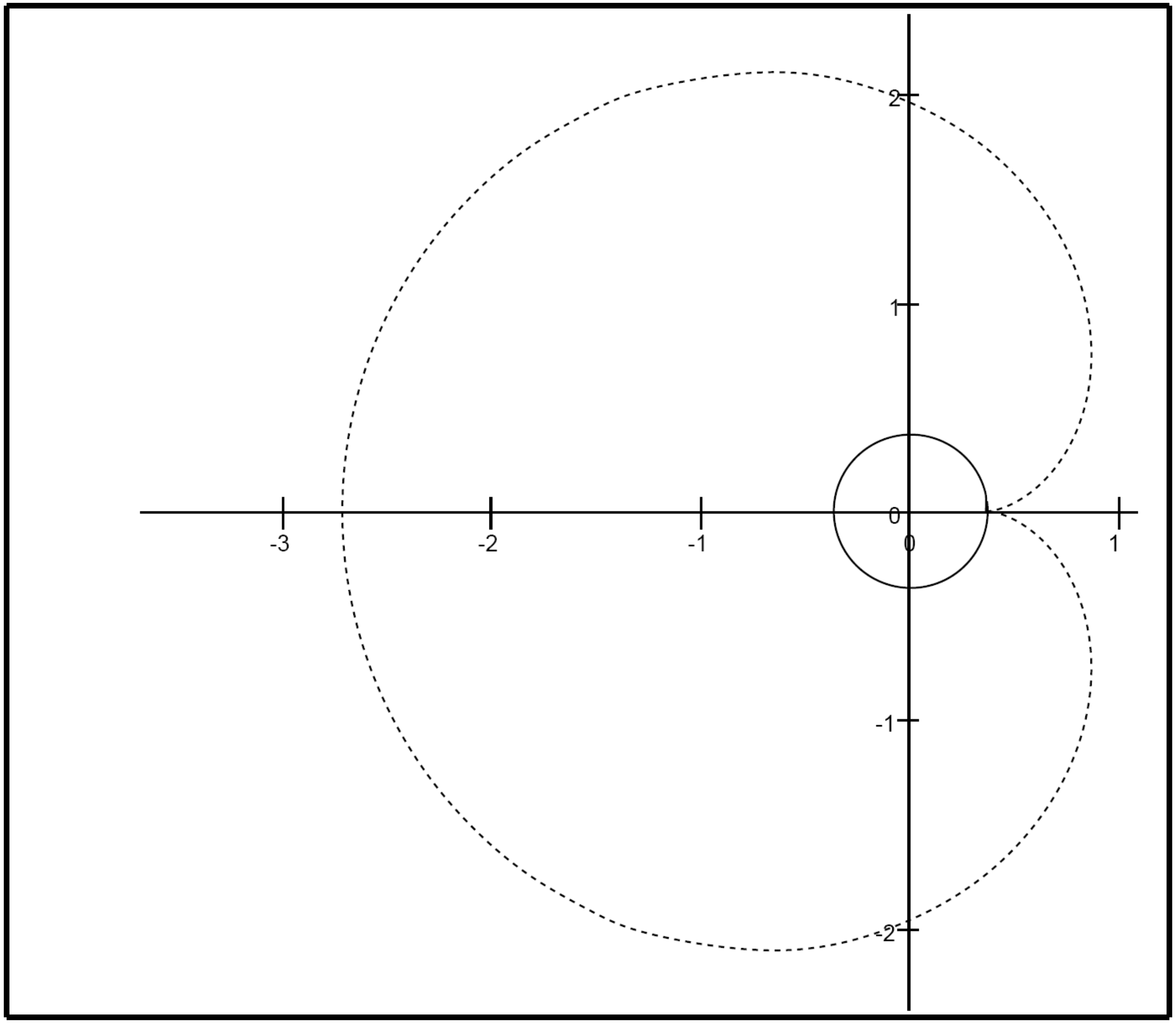
$$c_2 = a_0 a_1 a_2 + \frac{1}{2} a_0 a_1^2,$$

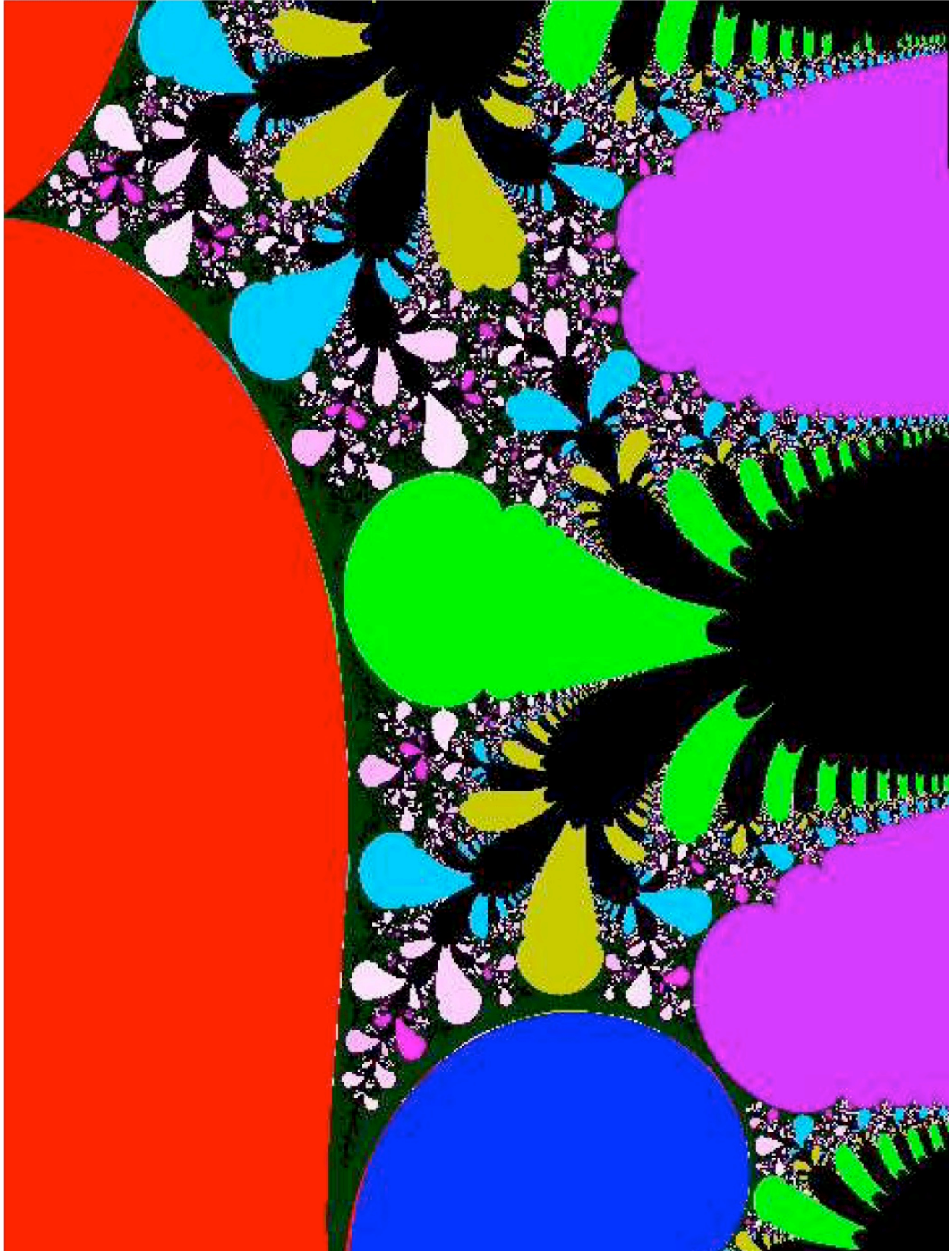
$$c_3 = a_0 a_1 a_2 a_3 + \frac{1}{2} a_0 a_1 a_2^2 + a_0 a_1^2 a_2 + \frac{1}{6} a_0 a_1^3$$

Example

$$e^{ze^{ze^{ze^{z\cdots}}}} = \sum_{n=0}^{\infty} \frac{(n+1)^{n-1}}{n!} z^n$$

Region of convergence





Continued fractions --- An IQ test

What's the next number in this sequence?

O, T, T, F, F, S, S, E, ... ??

14, 34, 42, 72, … ??

1, 5, 61, … ??

$$a_1 = b_1$$

$$a_2 = b_1 \left(b_1 + b_2\right)$$

$$a_3 = b_1 \left[b_2 b_3 + \left(b_1 + b_2\right)^2\right]$$

$$a_4 = b_1 \left[b_2 b_3 b_4 + b_2 b_3^2 + 2 b_2^2 b_3 + 2 b_1 b_2 b_3 + \left(b_1 + b_2\right)^3\right]$$

Padé Summation...

Table 8.10 Same as in Table 8.9 except that the approximants are evaluated at $x = 5$

n	Taylor series $\sum_{k=0}^{n} \frac{5^k}{k!}$	Padé approximants $P_M^N(5)$	Relative errors: Taylor series	Relative errors: Padé approximants
7	128.619	$P_4^3 =$ 71.385	−1.33 (−1)	−5.19 (−1)
8	138.307	$P_4^4 =$ 128.619	−6.81 (−2)	−1.33 (−1)
9	143.689	$P_5^4 =$ 158.621	−3.18 (−2)	6.88 (−2)
10	146.381	$P_5^5 =$ 149.697	−1.37 (−2)	8.65 (−3)
11	147.604	$P_6^5 =$ 148.001	−5.45 (−3)	−2.78 (−3)
12	148.114	$P_6^6 =$ 148.362	−2.02 (−3)	−3.43 (−4)
13	148.310	$P_7^6 =$ 148.427	−6.98 (−4)	9.05 (−5)
14	148.380	$P_7^7 =$ 148.415	−2.26 (−4)	1.03 (−5)
15	148.403	$P_8^7 =$ 148.413	−6.90 (−5)	−2.28 (−6)
16	148.410	$P_8^8 =$ 148.413	−1.99 (−5)	−2.41 (−7)
17	148.412	$P_9^8 =$ 148.143	−5.42 (−6)	4.56 (−8)
18	148.413	$P_9^9 =$ 148.413	−1.40 (−6)	4.48 (−9)
19	148.413	$P_{10}^9 =$ 148.413	−3.45 (−7)	−7.43 (−10)
20	148.413	$P_{10}^{10} =$ 148.413	−8.11 (−8)	−6.81 (−11)
21	148.413	$P_{11}^{10} =$ 148.413	−1.82 (−8)	1.00 (−11)
22	148.413	$P_{11}^{11} =$ 148.413	−3.91 (−9)	8.58 (−13)
23	148.413	$P_{12}^{11} =$ 148.413	−8.07 (−10)	−1.13 (−13)
24	148.413	$P_{12}^{12} =$ 148.413	−1.60 (−10)	−9.13 (−15)
25	148.413	$P_{13}^{12} =$ 148.413	−3.05 (−11)	1.09 (−15)
26	148.413	$P_{13}^{13} =$ 148.413	−5.60 (−12)	8.29 (−17)

Table 8.13 Padé summation of the Taylor series for $z^{-1} \ln(1+z)$ about $z = 0$

In Sec. 8.5, it will be shown that the sequences of Padé approximants $P_N^N(z)$ and $P_{N+1}^N(z)$ converge rapidly, even beyond the circle of convergence of the Taylor series $|z| < 1$. Observe that for real positive x the Padé approximants $P_N^N(x)$ monotonically decrease and $P_{N+1}^N(x)$ monotonically increase with N to the common limit $\ln(1+x)/x$. Thus, for any N, these Padé approximants supply upper and lower bounds on $\ln(1+x)/x$

$P_N^N(x)$

N	$x = 0.5$	$x = 1$	$x = 2$
1	0.812 500 000 0	0.700 000 000 0	0.571 428 571 4
2	0.810 945 273 6	0.693 333 333 3	0.550 724 637 7
3	0.810 930 365 3	0.693 152 454 8	0.549 402 823 0
4	0.810 930 217 7	0.693 147 332 4	0.549 312 879 5
5	0.810 930 216 2	0.693 147 185 0	0.549 306 618 4
6	0.810 930 216 2	0.693 147 180 7	0.549 306 177 9
7	0.810 930 216 2	0.693 147 180 6	0.549 306 146 7
8	0.810 930 216 2	0.693 147 180 6	0.549 306 144 5

$P_{N+1}^N(x)$

N	$x = 0.5$	$x = 1$	$x = 2$
1	0.810 810 810 8	0.692 307 692 3	0.545 454 545 5
2	0.810 928 961 7	0.693 121 693 1	0.549 019 607 8
3	0.810 930 203 2	0.693 146 417 4	0.549 285 176 8
4	0.810 930 216 1	0.693 147 157 9	0.549 304 620 9
5	0.810 930 216 2	0.693 147 179 9	0.549 306 034 1
6	0.810 930 216 2	0.693 147 180 5	0.549 306 136 4
7	0.810 930 216 2	0.693 147 180 6	0.549 306 143 8
8	0.810 930 216 2	0.693 147 180 6	0.549 306 144 3

Table 8.15 Padé approximants $P_N^N(x)$ and $P_{N+1}^N(x)$ for the Stieltjes series $\sum_{n=0}^{\infty} (-x)^n n!$ evaluated at $x = 1$ and $x = 10$

In contrast to these good results, the optimal asymptotic approximation to the Stieltjes series at $x = 1$ and $x = 10$ is worthless (the optimal truncation contains no terms of the series). On the other hand, $P_{14}^{14}(1)$ and $P_{15}^{14}(1)$ agree with the "sum" of the series, $\int_0^\infty e^{-t}/(1 + t)\, dt$, to better than five decimal places, while $P_{50}^{50}(1)$ and $P_{51}^{50}(1)$ agree to ten decimal places

	$x = 1$		$x = 10$	
N	$P_N^N(1)$	$P_{N+1}^N(1)$	$P_N^N(10)$	$P_{N+1}^N(10)$
0	1.0	0.5	1.0	0.090 91
1	0.666 67	0.571 43	0.523 81	0.128 63
2	0.615 38	0.588 24	0.379 73	0.149 66
3	0.602 74	0.593 30	0.314 24	0.162 95
4	0.598 80	0.595 08	0.278 47	0.171 96
5	0.597 38	0.595 78	0.256 73	0.178 36
6	0.596 82	0.596 08	0.242 56	0.183 06
7	0.596 57	0.596 21	0.232 84	0.186 60
8	0.596 46	0.596 28	0.225 93	0.189 32
9	0.596 41	0.596 31	0.220 86	0.191 45
10	0.596 38	0.596 33	0.217 06	0.193 13
50	0.596 35	0.596 35	0.201 56	0.201 39
∞	0.596 35	0.596 35	0.201 46	0.201 46

Padé summation of anharmonic oscillator ground-state energy series

$$E(\varepsilon) = 1/2 + 3/4\ \varepsilon - 21/8\ \varepsilon^2 + 333/16\ \varepsilon^3$$
$$- 30885/128\ \varepsilon^4 + 916731/256\ \varepsilon^5$$
$$- 65518401/1024\ \varepsilon^6 + 2723294673/2048\ \varepsilon^7$$
$$- 1030495099053/32768\ \varepsilon^8$$
$$+ 54626982511455/65536\ \varepsilon^9 - \ldots$$

Note that $E(1) = 8.03362$ e+8 (!!)

But…

Padé(m,n)	
(0,1)	0.66667
(1,1)	0.95600
(1,2)	0.73385
(2,2)	0.87411
(2,3)	0.76506
(3,3)	0.84110
(3,4)	0.78102
(4,4)	0.82529

Why do perturbation series diverge?

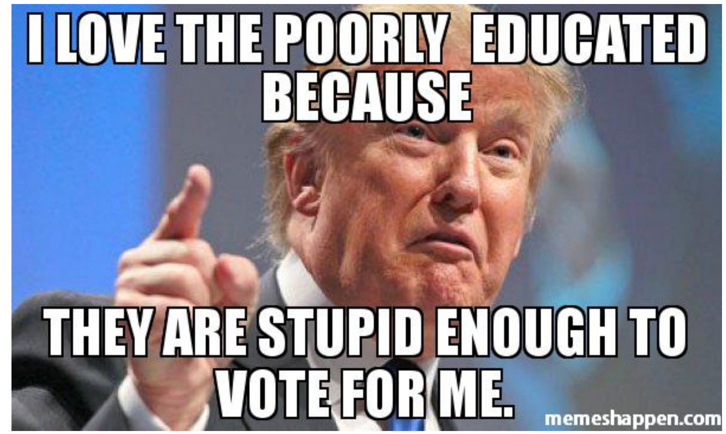
Mathematical reason **that perturbation series diverge --- many many Feynman graphs!**

Physical reason **that perturbation series diverge ---** ***level crossing***

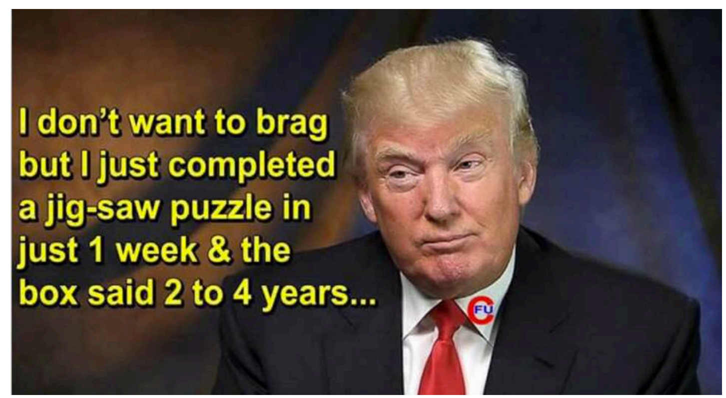
We must use complex analysis, which explains many things about the real world

The conventional world is described by *real* numbers:

-Election results



-IQ test results



-Money

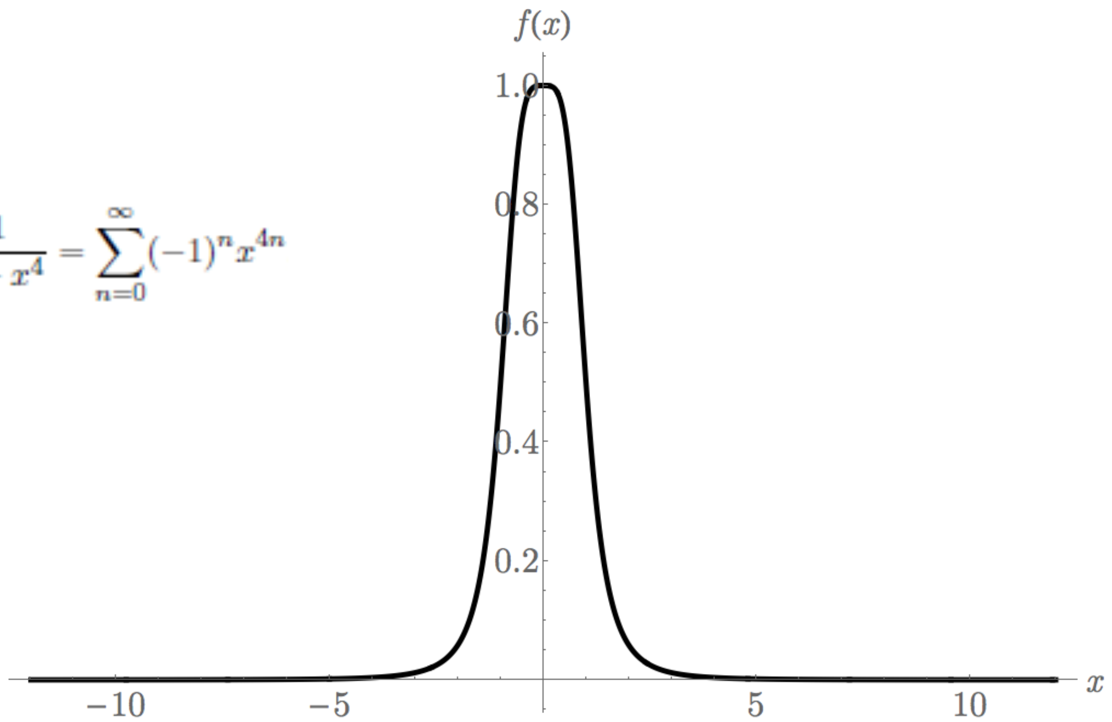
Real quantity of money …

Real quantity of money:

What can we learn from complex-variable heory?

(1) Why do Taylor series stop converging?

$$f(x) = \frac{1}{1+x^4} = \sum_{n=0}^{\infty} (-1)^n x^{4n}$$



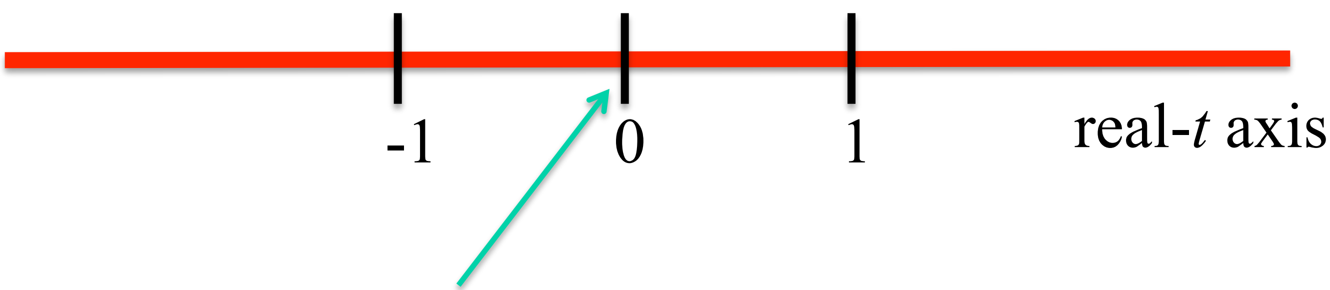
Answer: Singularities *in the complex plane*

Complex plane …

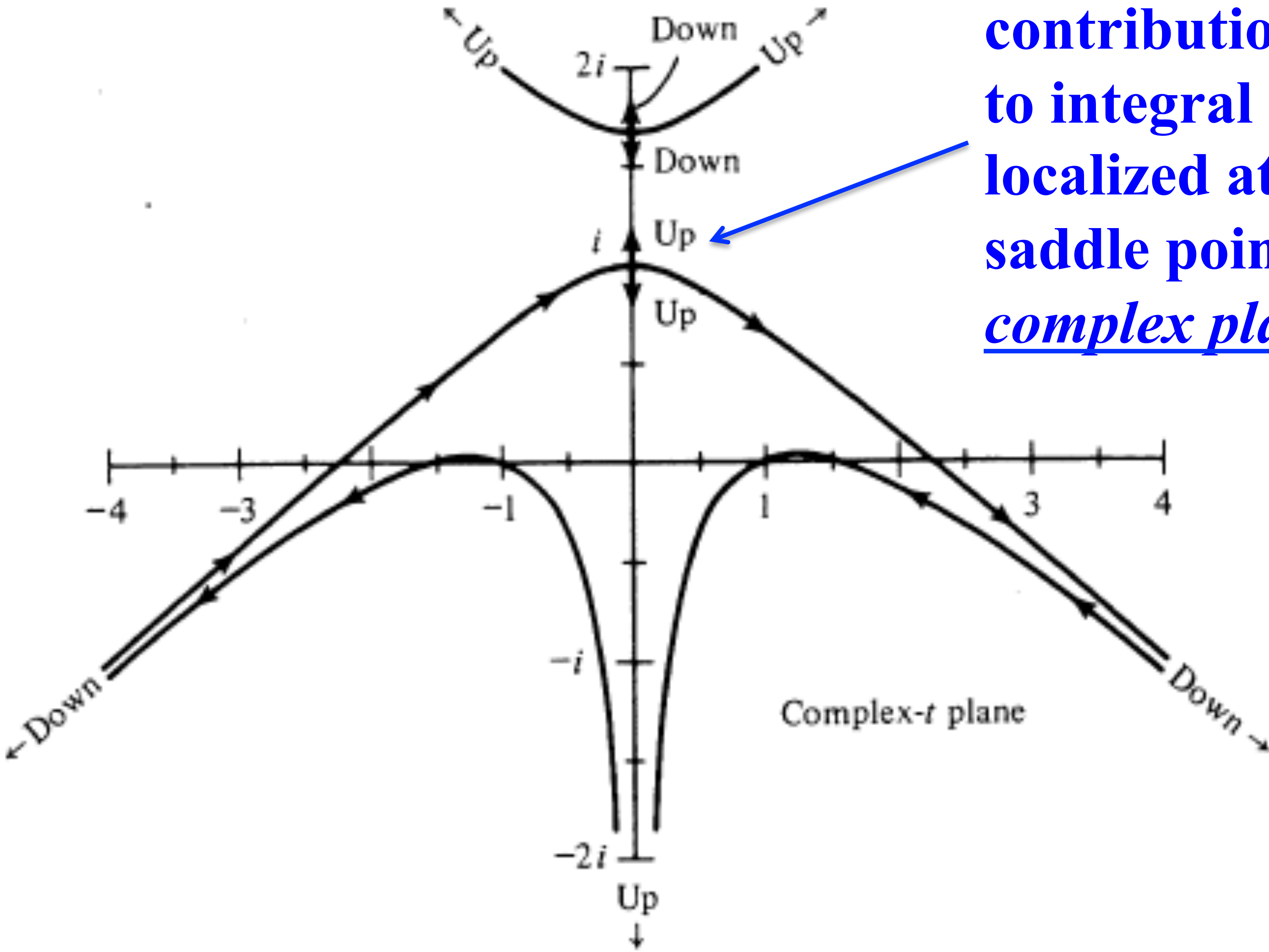
(2) Asymptotic expansion of integrals

$$I(x) = \int_{-1}^{1} dt\, e^{-4xt^2} \cos\left(5xt - xt^3\right)$$

$$I(x) \sim e^{-2x}\sqrt{\pi/x} \qquad (x \to +\infty)$$



For large x the contribution to
$I(x)$ is ***NOT*** localized at $t = 0$ (!!!)



For large |*x*| contribution to integral is localized at a saddle point in *complex plane*

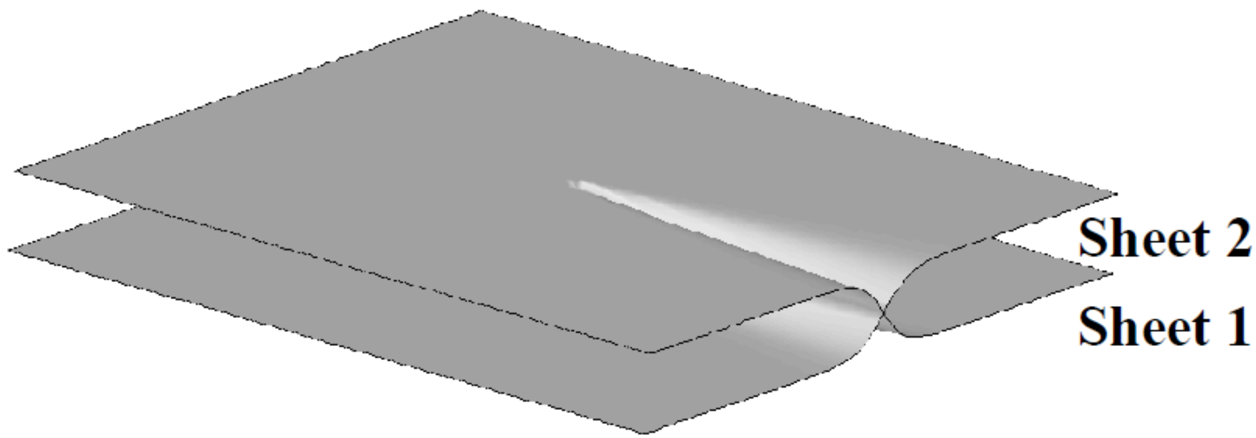
(3) Understanding real functions

Square-root function is *confusing*!

Why are there *two* answers??!

The square-root function must be defined on a *Riemann surface*…

Square-root function is defined on a two-sheeted *complex* Riemann surface:



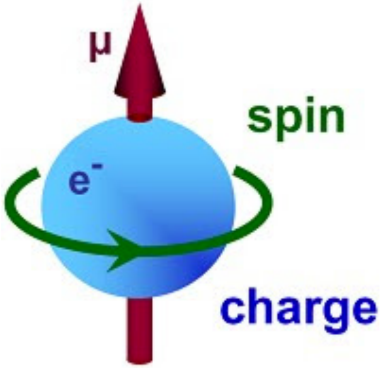
Riemann surface: two **complex planes cut and glued together**

If you go around *twice*, you return to the starting point…

Things that are the same when you go around *twice*…

Möbius strip…

Electron…



Modern physics already uses complex numbers:

Heisenberg algebra $xp - px = i\hbar$

Schrödinger equation

Eugene Wigner

Time reversal corresponds to complex conjugation --- changes the sign of i

**Quantum mechanics:
A particle in a potential well has *quantized* energy levels**

Going from one level to another is a "quantum leap"

(4) Complex analysis gives a deeper understanding of quantization…

Try to imagine a two-state system having energies *a* and *b*…

$$H = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

Introduce a *coupling constant* to couple the states:

$$H = \begin{pmatrix} a & g \\ g & b \end{pmatrix}$$

Energies for this two-state system

$$\det \begin{pmatrix} a - E & g \\ g & b - E \end{pmatrix} = 0$$

$$E^2 - (a+b)E + ab - g^2 = 0$$

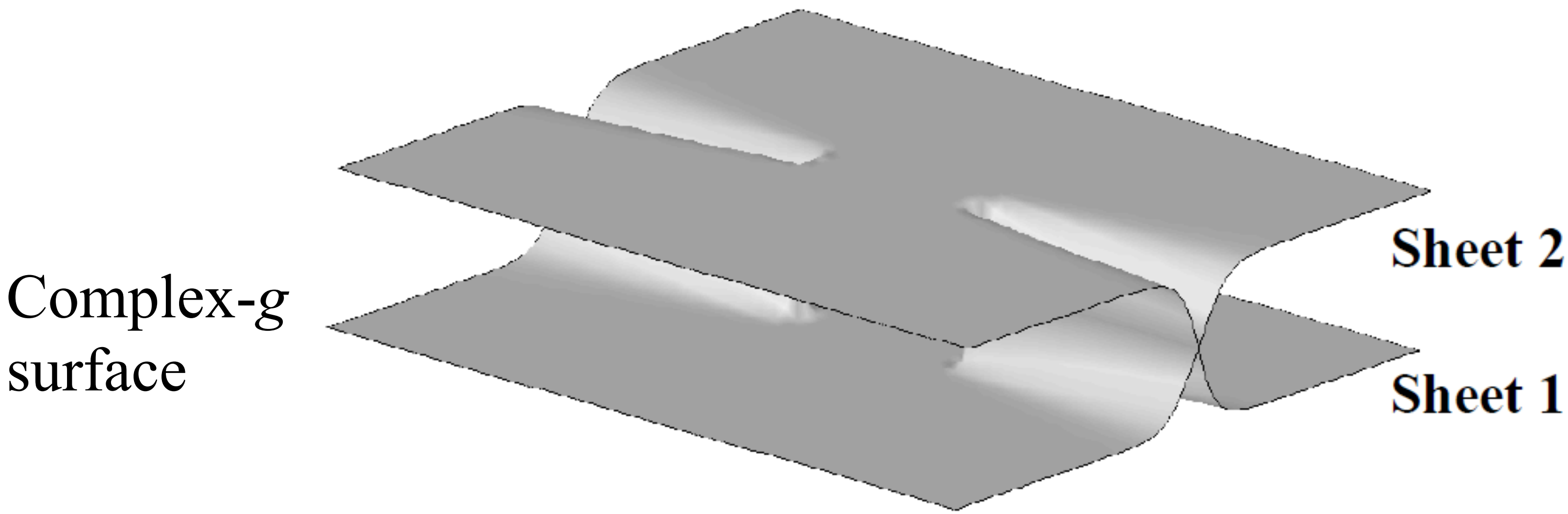
$$E(g) = \frac{a+b}{2} \pm \frac{1}{2}\sqrt{(a-b)^2 + 4g^2}$$

Level crossing occurs at square-root singularities in the complex-*g* plane at

$$g = \pm \frac{|a-b|}{2} i$$

(called *Bender-Wu singularities*)

$E(g)$ is defined on a **two-sheeted Riemann surface**:

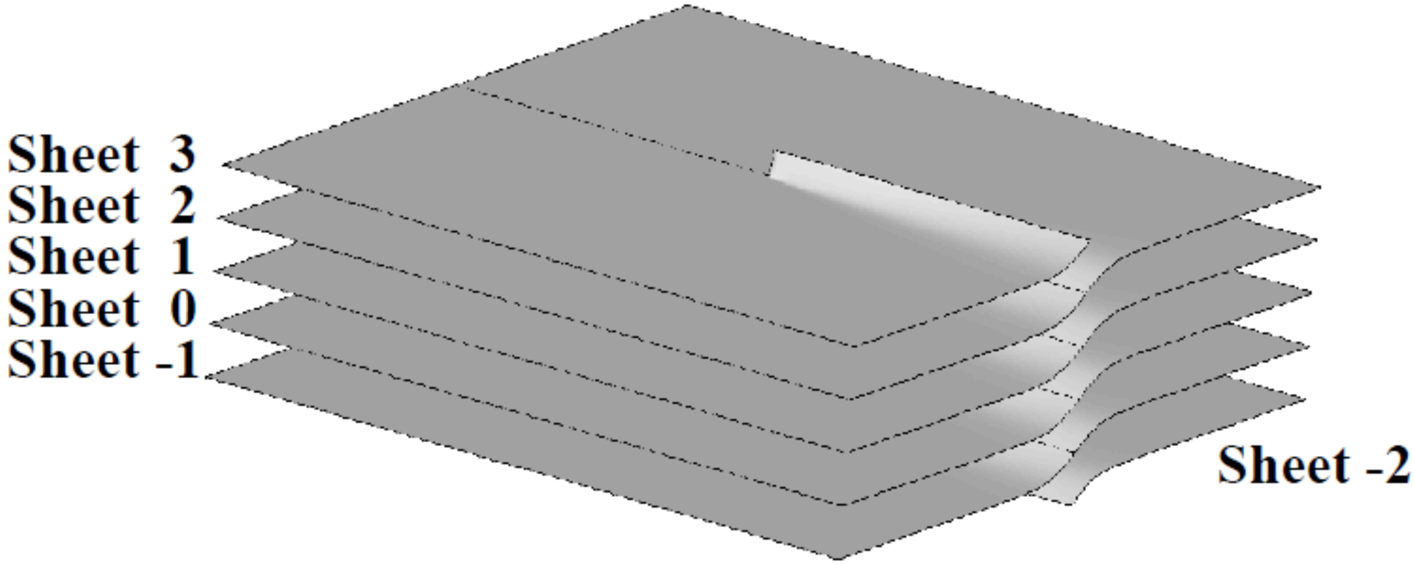


On the complex-g surface the quantum levels are ***not*** discrete

Quantization is *topological* – quantum energy levels correspond to the counting of sheets in a Riemann surface

These singularities explain the divergence of perturbation series (and complex-variable techniques are used to *sum* the series!)

Imagine a parking garage…



Energy levels are smooth analytic continuations of one another!

Laboratory analytic continuation of eigenvalues

(1) PRL **108**, 024101 (2012) PHYSICAL REVIEW LETTERS week ending 13 JANUARY 2012

$\mathcal{PT}$ Symmetry and Spontaneous Symmetry Breaking in a Microwave Billiard

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(Received 21 July 2011; published 10 January 2012)

We demonstrate the presence of parity-time ($\mathcal{PT}$) symmetry for the non-Hermitian two-state Hamiltonian of a dissipative microwave billiard in the vicinity of an exceptional point (EP). The shape of the billiard depends on two parameters. The Hamiltonian is determined from the measured resonance spectrum on a fine grid in the parameter plane. After applying a purely imaginary diagonal shift to the Hamiltonian, its eigenvalues are either real or complex conjugate on a curve, which passes through the EP. An appropriate basis choice reveals its $\mathcal{PT}$ symmetry. Spontaneous symmetry breaking occurs at the EP.

DOI: 10.1103/PhysRevLett.108.024101 PACS numbers: 05.45.Mt, 02.10.Yn, 11.30.Er

(2) H. Xu, D. Mason, L. Jiang, and J. G. E. Harris, *Nature* **537**, 80 (2016)

(3) J. Doppler, A. A. Mailybaev, J. Böhm, U. Kuhl, A. Girschik, F. Libisch, T. J. Milburn, P. Rabl, N. Moiseyev, and S. Rotter, *Nature* **537**, 76 (2016)

Note the term ***PT symmetry*** …

This brings us to topic #3 in this course −

***PT*-symmetric quantum theory**

PT-symmetric Hamiltonians are *complex deformations* of Hermitian Hamiltonians

You begin with a Hermitian Hamiltonian and introduce a *deformation parameter*…

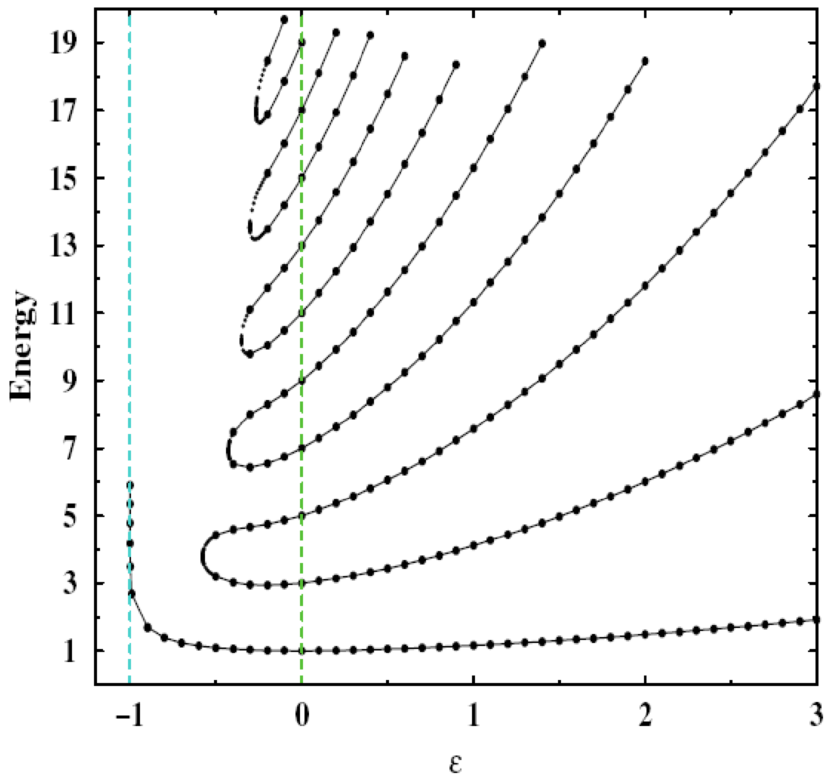
Complex deformed squirrel

Complex deformed frog

Complex deformed **parrot**

One-parameter family of *PT*-symmetric Hamiltonians obtained by complex deformation of the harmonic oscillator

$$H = p^2 + x^2(ix)^\varepsilon \quad (\varepsilon \text{ real})$$



Look! *H* is not Hermitian but its eigenvalues are **all real**!

Special cases: $H = p^2 + ix^3$ (cubic: ε=1) $H = p^2 - x^4$ (quartic: ε=2) $H = p^2 + x^6$ (sextic: ε=4)

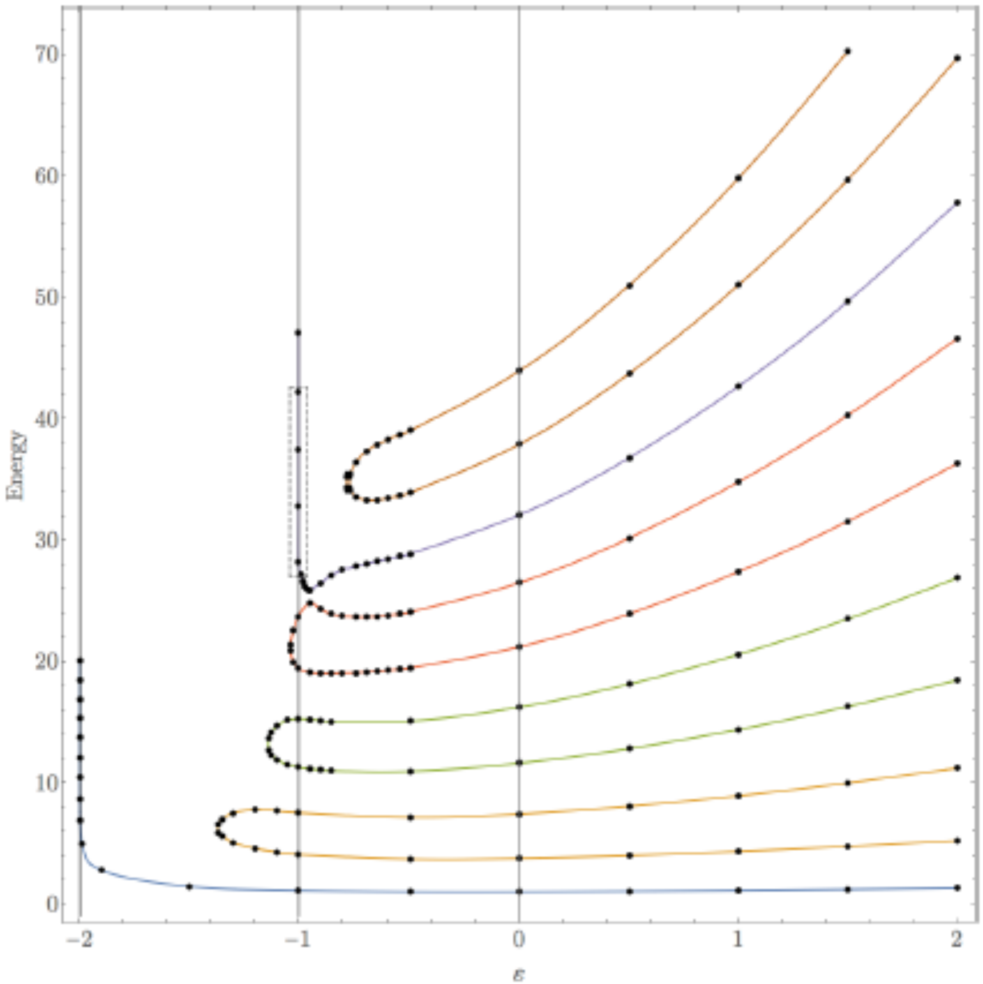
<u>Simple example:</u> $H = p^2 + x^2 + i\varepsilon x$

$$-\psi''(x) + x^2\psi(x) + i\varepsilon x\psi(x) = E\psi(x)$$

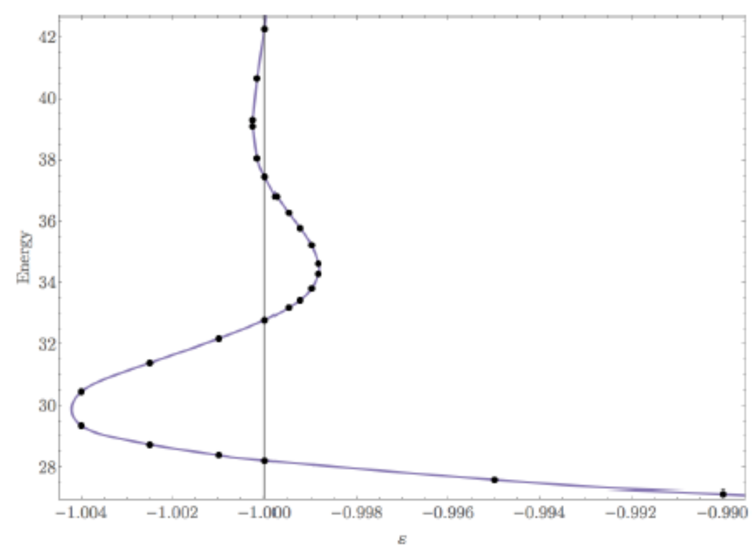
$$\psi(\pm\infty) = 0$$

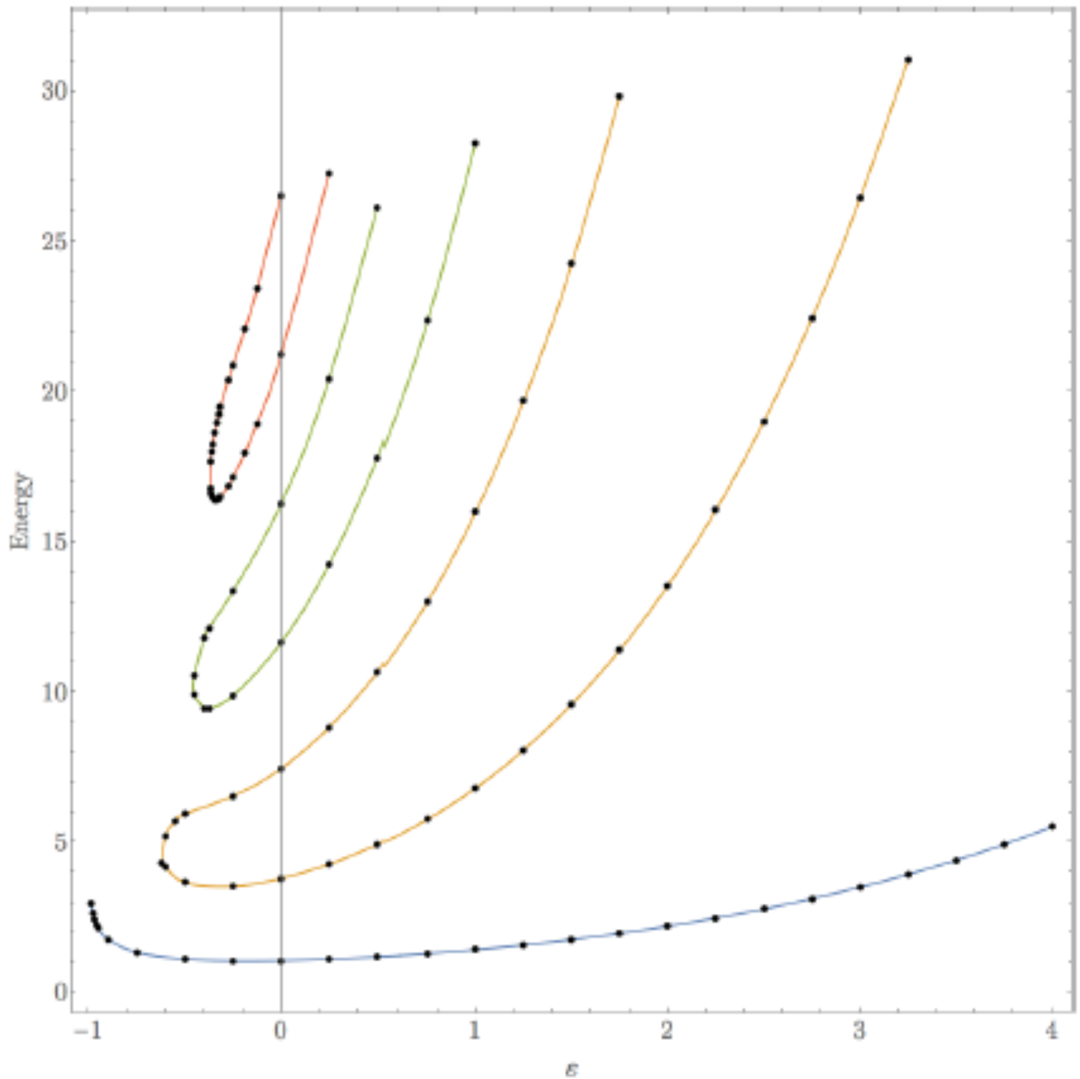
$E_n = 2n + 1 + \varepsilon^2/4$ $(n = 0, 1, 2, 3, \ldots)$

The picture is generic…

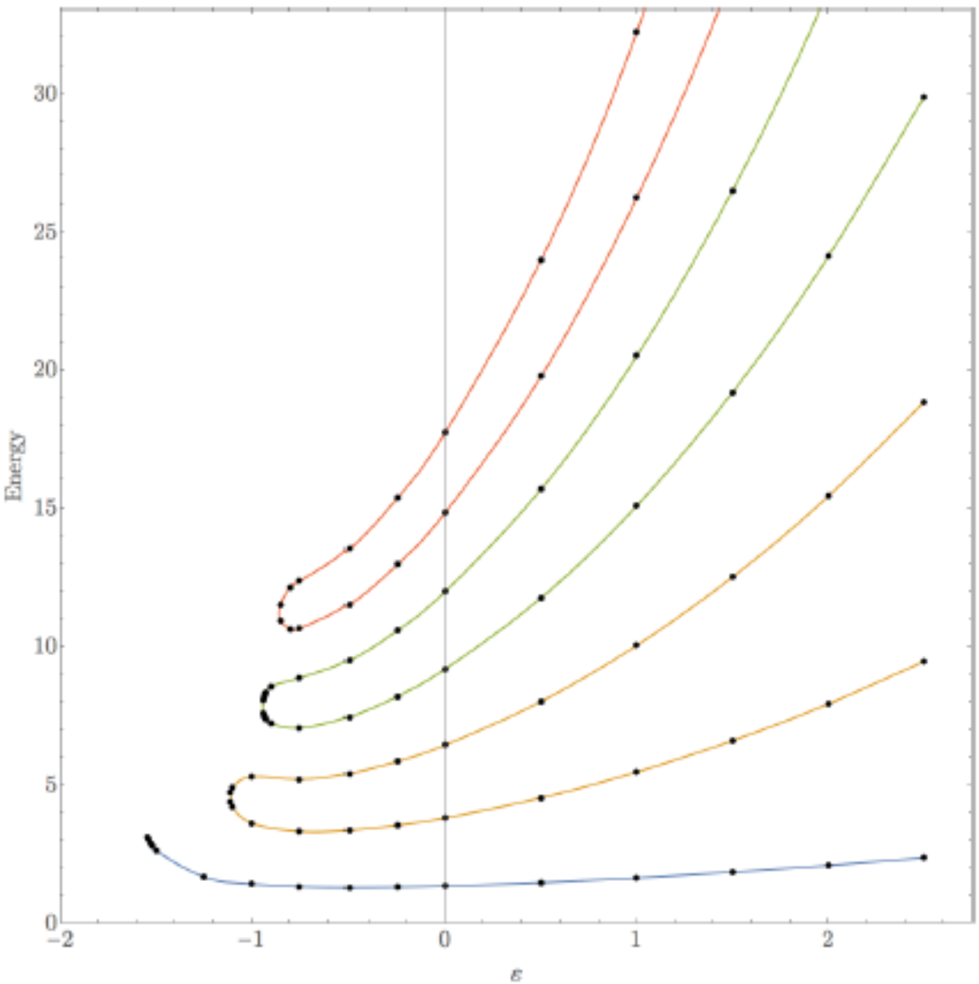


$$H = p^2 + x^4\ (ix)^{\varepsilon}$$





$$H = p^4 + x^2\,(ix)^{\varepsilon}$$



$$H = p^2 + x^2\ (ix)^\varepsilon \log(ix)$$

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First direct detection of gravitational waves

PT symmetry in quantum physics

EPL for the IYL 2015

Delicious ice cream: why does salt thaw ice?

Fascinating optics in a glass of water

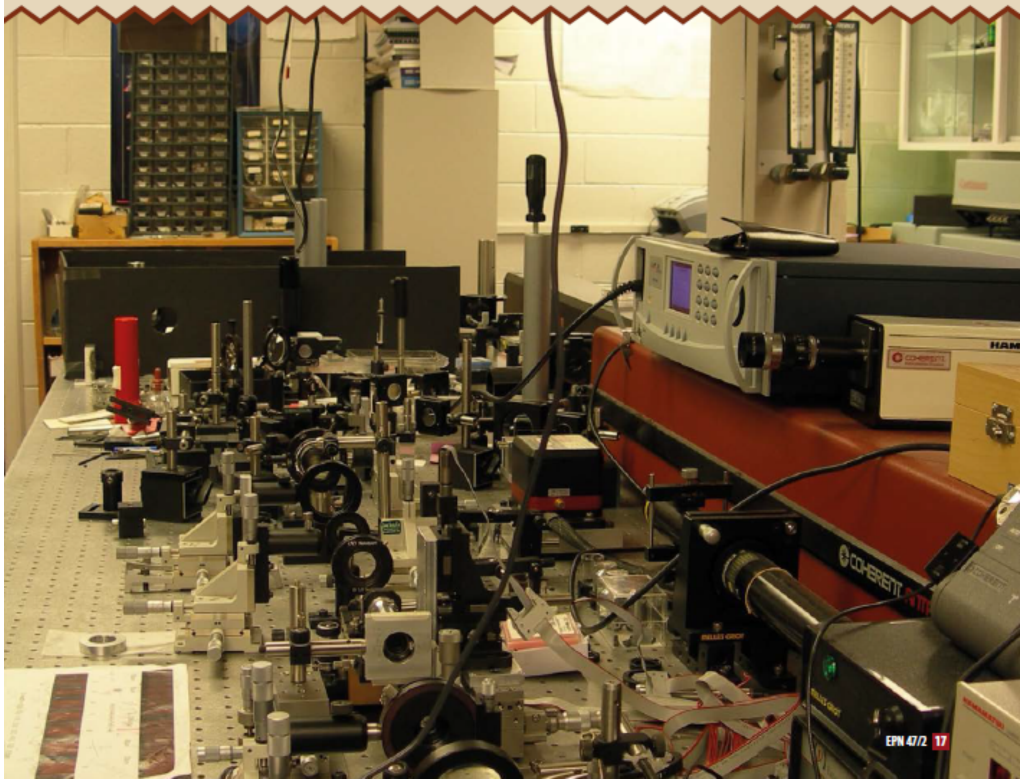
EPS

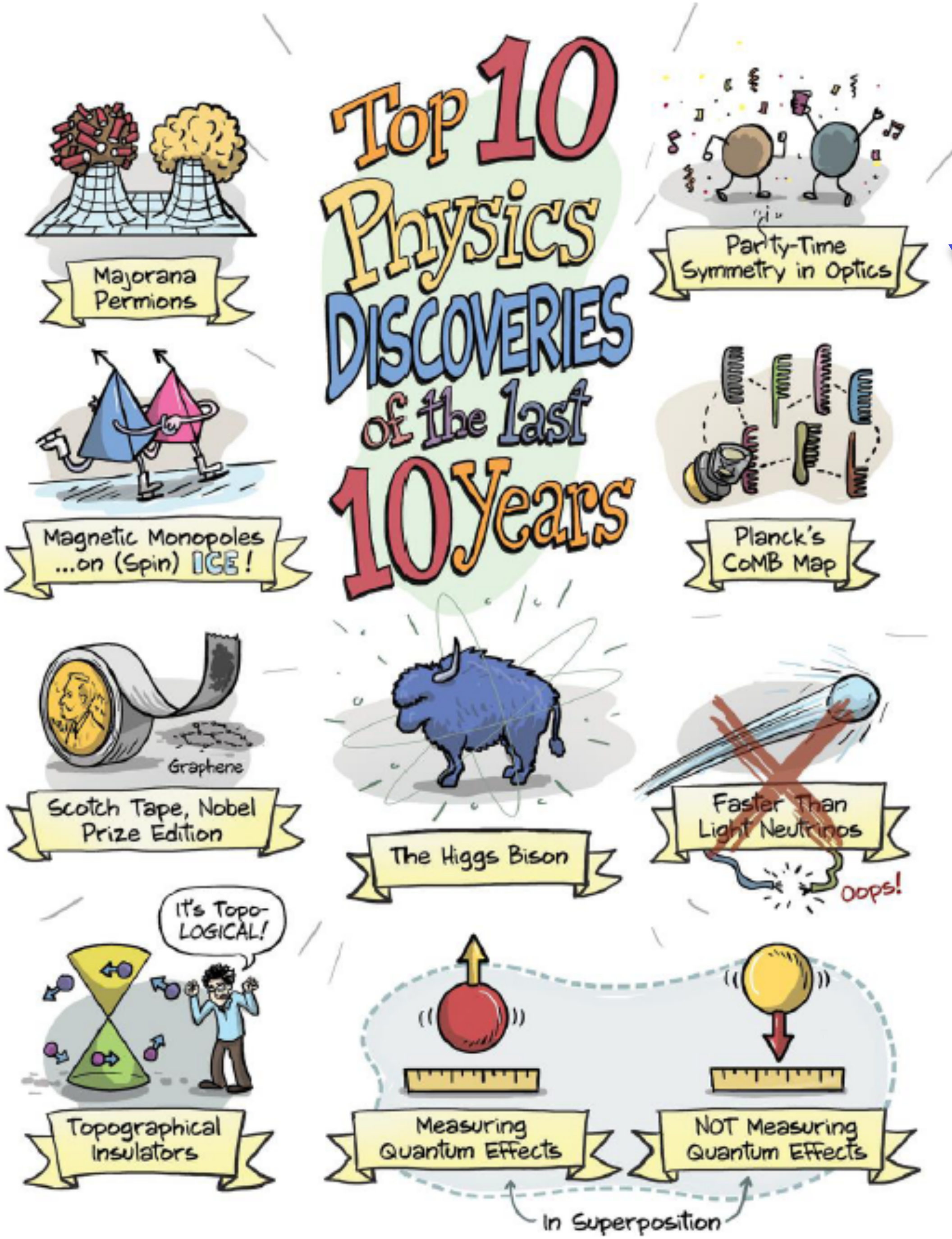
edp sciences

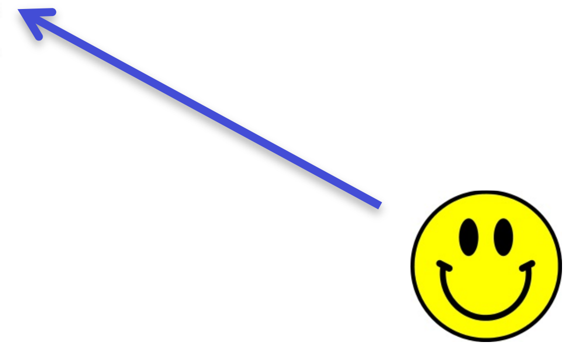
PT SYMMETRY IN QUANTUM PHYSICS: FROM A MATHEMATICAL CURIOSITY TO OPTICAL EXPERIMENTS

Carl M. Bender – Washington University in St. Louis, St. Louis, MO 63130, USA – DOI: http://dx.doi.org/10.1051/epn/2016201

Space-time reflection symmetry, or _PT symmetry_, first proposed in quantum mechanics by Bender and Boettcher in 1998 [1], has become an active research area in fundamental physics. More than two thousand papers have been published on the subject and papers have appeared in two dozen categories of the _arXiv_. Over two dozen international conferences and symposia specifically devoted to PT symmetry have been held and many PhD theses have been written.







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